Introduction to Quantum Field Theory

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March 3, 2011

Preface

These are notes made by a graduate student for graduate and undergraduate students. The intention is purely educational. They are a review of one the most beautiful fields on Physics and Mathematics, the Quantum Field Theory, and its mathematical extension, Topological Field Theories. The status of *review* is necessary to make it clear that one who wants to learn quantum field theory in a serious way should understand that she/he is not only required to read one book or review. Rather, it is important to keep studies on many classical books and their different approaches, and recent publications as well. Quantum field theories, together with topological field theories, are fields in evolution, with uncountable applications and uncountable approaches of learning it.

The idea of these notes initially started during my first year at Stony Brook University, when I was very well exposed to the subject, during the courses taught by Dr. George Sterman, [STERMAN1993], and by Dr. Dmitri Kharzeev, [KHARZEEV2010] . However, most of the first part of these notes was studies from classical books, mainly [PS1995], [SREDNICKI2007], [STERMAN1993], [WEINBERG2005], [ZEE2003]. This is just a tasting of a huge and intense field. In the continuation of the journey, I'm working on some derivations on topological quantum field theories, from classical refereces and books such as [IVANCEVIC2008], [LM2005], [DK2007], and the pioneering work of Edward Witten, [WITTEN1982], [WITTEN1988], [WITTEN1989], and [WITTEN1998-2].

I have divided this book into two parts. The first part is the *old-school* (and necessary) way of learning quantum field theory, and I shall call this section *Fundamentals of Quantum Field Theory*. In this part, in the first three chapters I write about scalar fields, fields with spin, and non-abelian fields. The following chapters are dedicated to *quantum electrodynamics* and *quantum chromodynamics*, followed by the *renormalization* theory.

The second part is dedicated to *Topological Field Theories*. A topological quantum field theory (TQFT) is a metric independent quantum field theory that introduces topological invariants of the background manifold. The best known example of a three-dimensional TQFT is the Chern-Simons-Witten theory. In these notes I start with an introduction of the mathematical formalism and the algebraic structure and axioms. The following chapters are the introduction of path integral and non-abelian theories in the new formalism. The last chapters are reserved to the three-dimensional Chern-Simons-Witten theory and the four-dimensional topological gauge theory and invariants of four-manifolds (the Donaldson and Seiberg-Witten theories).

I do not believe it is possible to ever finish this book, and probably this is exactly the fun about it. One property of Science is that there is always more to learn, more to think and more to discovery. That's what makes it so delightful! I conclude this preface citing Dr. Mark Srednick on [SREDNICKI2007],

> `You are about to embark on  tour  of one of humanity's greatest intellectual endeavors, and certainty the one that has produced the most precise and accurate description of the natural world as we find it. I hope you enjoy your ride.`

Acknowledgment

These notes were made during my work as a PhD student at State University of New York at Stony Brook, under the support of teaching assistant, and later of research assistent, for the department of Physics of this university. I'd like to thank Prof. Dima Kharzeev, Prof. George Sterman, Prof. Patrick Meade, Prof. Sasha Kirillov, Prof. Dima Averin, and Prof. Barbara Jacak, for excellent courses and discussions, allowing me to have the basic understanding to start this project.

Marina von Steinkirch,
January of 2011.

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Part I

Fundamentals of Quantum Field Theory

Chapter 1

Spin Zero

In this first chapter we will begin studying *scalar fields*, i.e. particles with no spin. They are the simplest way of getting the first techniques of quantum field theory, even though there is no known scalar particles in nature.[1]

Before we start our journey in quantum field theory, I would like to say two important things. One of them is that it is always useful to perform dimensional analysis on our Lagrangians, operators, etc. In the natural units, where

$$[l] = [t] = [m]^{-1} = [E]^{-1},$$

we have $[\mathcal{L}] = E^4$.

When we are working on phenomenological problems, it is also useful to remember that 200 MeV $\sim$ 1 fm^{-1}.

1.1 Quantization of the Point Particle

Supposing a particle with only a defined momentum, with no charges and no spin, such as the Dirac's original photon. The recipe of quantization from a classical picture is:

1. Start with a coordinate $q(t)$, and the classical Lagrangian $L(q(t), \dot{q}(t))$.

2. Write the Hamiltonian $H_{cl}(p, q) = p\dot{q} - L_{cl}$, where p is the conjugate momentum.

3. Postulate $H(\hat{p}, \hat{q})\psi = -i\hbar\frac{\partial}{\partial q}\psi$.

[1]There are theoretical candidates for scalar fields in nature, such as some models including the *Higgs particle*.

4. For the nonrelativistic case, the Hamiltonian of the free particle is $H = \frac{p^2}{2m}$, for the relativistic case it is $H = c\sqrt{p^2 + (mc)^2}$.

In the same logic one can restrict the global field to a local field theory on x, writing the Lagrangian as

$$L(t) = \int d^3x \mathcal{L}\Big(\phi_a(x,t), d_\mu \phi_a(x,t)\Big).$$

The action is then

$$S_v = \int_{t_1}^{t_2} dt L(t) = \int_{t_1}^{t_2} dt \int_{V(t')} d^3x \mathcal{L}[\phi_a].$$

The *Principle of Minimum Action* says that any variation on the action should be zero

$$\delta S = 0.$$
$$\delta\phi(x,t_1) = \delta\phi(x,t_2) = 0,$$

and from performing this variation on the action,

$$\begin{aligned} 0 &= \int dtdx^3 \left(\frac{\partial \mathcal{L}}{\partial \phi_a} \delta\phi_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \mathcal{L}(\partial_\mu \phi_a) \right), \\ &= \int dtdx^3 \left(\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \right) \delta_a, \end{aligned}$$

one gets the *Lagrange's equations* of motion for this field,

$$\frac{\partial \mathcal{L}}{\partial \phi_i} - \frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{(\partial_\mu \phi_i)} = 0. \tag{1.1.1}$$

Example: The real Klein-Gordon Equation

The Klein-Gordon Lagrangian[3] is [2]

$$\mathcal{L} = \frac{1}{2}[(\partial^\mu \phi)(\partial_\mu \phi) - m^2\phi^2], \tag{1.1.2}$$

and by varying the action, one can find its equation of motion

$$(\Box + m^2)\phi = 0.$$

[2] This is the density of the Lagrangian, but since it is a common practice to call it just Lagrangian, we will follow this convention here.

Example: Complex Scalar Field

Treating the field and its complex conjugate as independent,

$$\phi = (\phi_1 + i\phi_2),$$

one can writes a Lagrangian as

$$\mathcal{L} = \partial^\mu \phi^* \partial_\mu \phi - m^2 \phi^* \phi. \tag{1.1.3}$$

1.2 Form Invariant Lagrangians

A transformation on the Lagrangian is relevant for quantum field theory when this transformation keeps the Lagrangian form invariant,

$$\bar{\mathcal{L}}(\bar{\phi}, \frac{\partial \bar{\phi}_i}{\partial y^\mu}) = \mathcal{L}(\phi_i(x), \frac{\partial \phi_i}{\partial x}) \frac{d^4 x}{d^4 y}. \tag{1.2.1}$$

$$\bar{\mathcal{L}} d^4 y = \bar{\mathcal{L}} d^4 x. \tag{1.2.2}$$

For these cases of transformations, solutions in the equation of motion on x implies solutions in y. The most general invariant transformation is giving by modifying equation (1.2.2) by any term that lives on the surface, for instance $\frac{dF^\mu}{dy^\mu} d^4 y$.

In the case of our local field, we can generalize the invariance studying the infinitesimal transformations of coordinates and fields. Introducing a set of parameters $\{\beta_\alpha\}_{\alpha=1}^N$, in terms of a small $\delta\beta_\alpha$:

$$\bar{x} = x^\mu + \delta x^\mu(\delta\beta_\alpha) = x^\mu + \sum_{\alpha=1}^{N} \frac{\partial(\delta x^\mu)}{\partial(\delta\beta_\alpha)} \delta\beta_\alpha, \tag{1.2.3}$$

$$\bar{\phi}_i = \phi_i(x) + \delta\phi_i(\delta\beta_\alpha) = \phi_i(x) + \sum_{\alpha=1}^{N} \frac{\partial(\delta\phi_i)}{\partial(\delta\beta_\alpha)} \delta\beta_\alpha. \tag{1.2.4}$$

One example it is the translation transformation where $\delta x^\mu = \delta a^\mu$ and one has $\frac{\partial(\delta x^\mu)}{\partial(\delta a^\mu)} = g^\mu_\nu = \delta_{\mu\nu}$.

1.3 Noether's Theorem

Every time we have a transformation on the Lagrangian that keeps it invariant, we can say that we have a *symmetry*. From the *Noether's Theorem*,

$\mathcal{L}(\phi_i, \delta_\mu \phi_i)$ forms N conserved currents and charges. Therefore, under some transformation with N parameters one has the conserved current

$$\partial_\mu J^\mu_a = 0, \tag{1.3.1}$$
$$J^\mu_a = -\mathcal{L}\frac{\partial(\delta x^\mu)}{\partial(\beta_a)} - \sum_{\phi_i} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i)} \frac{\partial(\delta_* \phi_i)}{\partial(\beta_a)}, \tag{1.3.2}$$

where a runs from 1 to N and β_a is the parameters of variation defined last section. To apply (2.7.1), one needs to understand the concept of variation at a point, which is the variation between two fields at a point, i.e.

$$\delta_* \phi_i = \bar{\phi}_i(x) - \phi_i(x).$$

The current can also be rewritten in the form of the *Energy-Moment Tensor*, using a more compact notation

$$J^\mu_a = -\mathcal{L}\delta^\mu_\alpha + \frac{\delta \mathcal{L}}{\partial(\partial_\mu \phi)} \partial_\alpha \phi = T^\mu_\alpha. \tag{1.3.3}$$

This tensor can be integrated on the space giving

$$P^\nu = \int d^3x T^{0\nu} = \int d^3x T^0_\lambda g^{\lambda\nu}. \tag{1.3.4}$$

To extract the important informations of this equation one separates the tensor in the temporal (energy) and spatial (momentum) parts. The energy is given by the zeroth component of (1.3.4):

$$P^0 = \int d^3x T^{00}.$$

Defining the conjugate momentum as

$$\Pi_i = \frac{\delta \mathcal{L}}{\partial \dot{\phi}_i} = \partial_0 \phi^*_i,$$

the other components are given by

$$P^i = \int d^3x [\sum_i \Pi_i \dot{\phi}^i - \mathcal{L}],$$

where the current vector is

$$\vec{P} = \int d^3x [\sum_i \Pi_i \nabla \phi_i].$$

A second conserved quantity is the angular momentum given by

$$L^{ij} = \int d^3x(x^i P^j - x^j P^i).$$

A third quantity that is conserved is the charge,

$$Q^i = -\int d^3x \Pi_i \frac{\partial \phi}{\partial x^i}.$$

A consequence of (1.2.4) is that any Lagrangian of a complex scalar fields is form invariant under a *phase* transformation, i.e. $\bar{\phi}(x) \to e^{i\sum \beta_a t_a}\phi(x)$. These are the transformations generated by the gauge group $U(1)$. If one starts with a defining representation R of this Lie group, the conserved current is

$$J_a^\mu = i\sum_{i,j=1}^{N}\left\{\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_i^*)}[t_a^R]_{ij}\phi_j^* - \text{ complex conjugate }\right\},$$

where t_a^R is the generators of this representation.

Example: The $U(1)$ transformation for the complex Klein-Gordon Equation

From (1.1.3), one can add a potential field of the kind $U(|\phi|^2)$,

$$\mathcal{L} = (\partial_\mu \phi)(\partial^\mu \phi^*) - m^2|\phi|^2 + U(|\phi|^2),$$

Writing $\bar{\phi}(x) = e^{i\theta}\phi(x)$, the parameter θ is conserved, giving one conserved current. This charge is exactly the electromagnetic charge when it couples to the field.

1.4 The Poincare Group

The *Poincare group* is the group of translations (a^μ) plus the Lorentz transformations (Λ_ν^μ),

$$\bar{x}^\mu = \Lambda_\nu^\mu x^\nu + a^\mu,$$

which representation is given by $D(a, \Lambda)$. The Lorentz transformations are defined by

$$\begin{aligned}\bar{x}^\mu &= \Lambda_\nu^\mu x^\nu,\\ \Lambda_\nu^\mu &= g_{\mu\alpha} g^{\nu\beta}\Lambda_\beta^\alpha,\\ \Lambda_\nu^\mu &= (\Lambda^{-1})_\nu^\mu.\end{aligned}$$

The six generators $(\delta\lambda)$ are found near the identity

$$\begin{aligned} \Lambda^{\mu}_{\nu} &= \delta^{\mu}_{\nu} + \delta\lambda^{\mu}_{\nu}, && (1.4.1) \\ &= (I + \delta\lambda)^{\mu}_{\nu}, && (1.4.2) \end{aligned}$$

where in general the determinant of Λ is 1. This matrix can be written explicitly for a representation with a rotating term and then a boost as

$$\Lambda^{\mu}_{\nu} = (e^{i\omega.k} e^{-i\theta.j})^{\mu}_{\nu}.$$

The generators of the Poincare group in the representation of the fields ϕ_b are

$$\begin{aligned} (\hat{p}_{\mu})_{ab} &= -i\delta_{ab}\partial_{\mu}. \\ (\hat{m}_{\mu\nu})_{ab} &= -i\delta_{ab}(x_{\mu}\partial_{\nu} - x_{\nu}\partial_{\mu}) - (\Sigma_{\mu\nu})_{ab}, \end{aligned}$$

with an algebra given by

$$[\hat{p}_{\mu}, \hat{p}_{\nu}] = 0,$$

$$[\hat{p}_{\mu}, \hat{m}_{\lambda\gamma}] = ig_{\mu\lambda}\hat{p}_{\sigma} - ig_{\mu\sigma}\hat{p}_{\lambda},$$

$$[\hat{m}_{\mu\nu}, \hat{m}_{\lambda\nu}] = ig_{\mu\lambda}\hat{m}_{\nu\sigma} + \text{ 3 terms.}$$

1.5 Quantization of Scalar Fields

To start the quantization of the fields ϕ and its conjugate momentum Π, one postulates the basic equal-time commutators (ETCRS),

$$[\phi(x, x_0), \Pi(y, x_0)] = i\hbar\delta^3(x - y), \quad (1.5.1)$$

$$[\phi^*(x, x_0), \Pi^*(y, x_0)] = -i\hbar\delta^3(x - y), \quad (1.5.2)$$

$$[\phi(x, x_0), \phi(y, x_0)] = [\Pi(x, x_0), \Pi(y, x_0)] = 0. \quad (1.5.3)$$

1.6 Transforming States and Fields

To see how one transforms states and fields, let us suppose the system in the frame F, with states $|\psi\rangle$. In the frame $\bar{F}$, with states $|\bar{\psi}\rangle$, one needs to find a unitary transformation $U^{-1} = U^T$ such as

$$|\bar{\psi}\rangle = U(F, F')|\psi\rangle, \quad (1.6.1)$$

which must preserve the norm $\langle\psi|\psi\rangle = 1$. The classical transformation of our fields is given by

$$\bar{\phi}_b(\bar{x}) = \delta_{ba}(\Lambda)\phi_a(x),$$

which in quantum mechanics means

$$\langle\bar{\psi}|\bar{\phi}_b(\bar{x})|\bar{\psi}\rangle = \delta_{ba}(\bar{F}, F)\langle\psi|\phi_k(x)|\psi\rangle.$$

Therefore, one rewrites ((1.6.1)) as

$$\begin{aligned} |\psi\rangle &= U^{-1}(F, F')|\bar{\psi}\rangle, \\ \langle\psi| &= \langle\bar{\psi}|U(F, F'), \end{aligned}$$

getting

$$\left\langle\bar{\psi}\middle|\{\bar{\phi}_b(\bar{x}) = \delta_{ab}(\Lambda)\phi_a(x)\}\middle|\bar{\psi}\right\rangle = \left\langle\psi\middle|U^{-1}\delta_{ab}(\Lambda)\phi_a(x)U\middle|\psi\right\rangle.$$

Example: Translation as a Unitary Transformation

In a transformation such as translation, the unitary matrix is given by

$$U(a) = e^{ia_\mu p^\mu(x)}, \tag{1.6.2}$$

$$U(a)\phi(x)U^{-1}(a) = \phi(x + a). \tag{1.6.3}$$

To see how (1.6.3) works, one can insert it as unity ($U^{-1}U = 1$), and see how translation invariance appears connecting the fields:

$$\langle p_2|\prod_{i=1}^{n}\phi(x_i)|p_1\rangle = e^{-i(p_1-p_2)a}\langle q_2|\prod_i \phi(x_i - a)|q_1\rangle.$$

The unitary translation operator, which we shall call $\mathcal{F}$, can then be seen as a spatial shift on the states

$$\begin{aligned} |x'\rangle &\to |x' + d\bar{x}\ \rangle, \\ \mathcal{F}(x') &= |x' + d\bar{x}\ \rangle, \\ \mathcal{F}(x') &= 1 - ik.dx, \\ &= 1 - \frac{i}{\hbar}p.dz, \end{aligned}$$

respecting the canonical commutations that we know from quantum mechanics,

$$[x_i, k_j] = i\delta_{ij} \text{ or } [x_i, p_j] = i\hbar\delta_{ij}.$$

1.7 Momentum Expansion of Fields

We have now our fields quantized and we can work on the momentum expansion of the fields. This involves their *Fourier transformations*

$$\tilde{\phi}(k, x_0) = \int d^3x e^{-ikx}\phi(x, x_0),$$
$$\tilde{\Pi}(k, x_0) = \int d^3x e^{-ikx}\Pi(x, x_0).$$

It is necessary to write our ϕ and Π in terms of the creation/annihilation operators (such as when we do it for x and p in quantum mechanics, defining the raising/lowering operators), as can be seen in (1.7.1),

$$\begin{aligned} a(k, x_0) &= \frac{1}{(2\pi)^{3/2}}\left(\omega_k\tilde{\phi}(k, x_0) + i\tilde{\Pi}(k, x_0)\right) \\ a^\dagger(k, x_0) &= \frac{1}{(2\pi)^{3/2}}\left(\omega_k\tilde{\phi}(k, x_0) - i\tilde{\Pi}(k, x_0)\right) \end{aligned} \tag{1.7.1}$$

where $\omega_k = \sqrt{k^2 + m^2}$, which is the on-shell energy equation[4] $E^2 = p^2 + m^2$. The four-vector notation can be written as $k^\mu = (\omega, \vec{k})$.

From the definition of the ETCR given by (1.5.1), (1.5.2) and (1.5.3), one can then computes the ETCRs for (1.7.1), [3]

$$[a(k, x_0), a^\dagger(k, x_0)] = 2\hbar\omega_k\delta^3(k - k'), \tag{1.7.2}$$

$$[a, a] = [a^\dagger, a^\dagger] = 0. \tag{1.7.3}$$

Using ((1.7.1)) to rewrite the fields in terms of the creation/annihilation operators, we have

$$\begin{aligned} \phi(x, x_0) &= \int \frac{d^3k}{(2\pi)^{3/2}2\omega_k}[a(k, x_0)e^{ikx} + a^\dagger(k, x_0)e^{-ikx}], \\ \Pi(x, x_0) &= -i\int \frac{d^3k}{(2\pi)^{3/2}2}[a(k, x_0)e^{ikx} - a^\dagger(k, x_0)e^{-ikx}]. \end{aligned} \tag{1.7.4}$$

Substituting (1.7.4) in the Hamiltonian for scalar fields, one gets

$$H = \frac{1}{4}\int d^3k[a(k, x_0)a^\dagger(k, x_0) + a^\dagger(k, x_0)a(k, x_0)]. \tag{1.7.5}$$

[3]Setting $c = 1$, the natural unit.

A remarkable characteristic of the free fields is the fact that the Lagrangian is quadratic in fields, as you can see in (1.7.5). Making use of the temporal evolution of the fields, one can calculate the commutation of the creation/annihilation operator with this Hamiltonian,

$$[H, a(k, k_0)] = -\frac{da(k, x_0)}{dt} = \omega_k a(k, x_0).$$

1.8 States and Fock Space

The *Fock Space* is defined by the kets $|\{k_1\}\rangle$, where all states are found by applying the raising/creation operators from the ground states. Considering

$$|E'\rangle = a^\dagger(k)|E\rangle,$$

one computes

$$\begin{aligned} H|E'\rangle &= Ha^\dagger(k)|E\rangle, \\ &= [H, a^\dagger(k)]|E\rangle + a^\dagger(k)H|E\rangle, \\ &= \omega_k a^\dagger(k)|E\rangle + Ea^\dagger(k)|E\rangle, \\ &= (E + \omega_k)|E'\rangle, \end{aligned}$$

which clearly shows the shift on the energy when one applies the Hamiltonian operator on some eigenstate. Requiring that the system has a ground state $|0\rangle$ allows

$$a(k)|0\rangle = 0,$$

Therefore, it is possible to define the following states as

$$|k_1, ..., k_n\rangle = \prod_{i=1}^{n} a^\dagger(k_i)|0\rangle,$$

and applying the Hamiltonian operator, one finally has

$$H|k_1, ..., k_n\rangle = \sum_{i=1} \omega(k_i)|k_1, ..., k_n\rangle.$$

The dimension of the space is given by $\mathcal{F} = \mathcal{F}_0 \oplus \mathcal{F}_1 \oplus ... \oplus \mathcal{F}_N$, where N in $\mathcal{F}$ is determined by the *number operator*,

$$N = \int \frac{d^3k}{2\omega_k} a^\dagger(k)a(k), \tag{1.8.1}$$

$$N\Big(\prod_{i=1}^{N} a^\dagger(k_i)|0\rangle\Big) = n\Big(\prod_{i=1}^{N} a^\dagger(k_i)|0\rangle\Big). \tag{1.8.2}$$

It is necessary to *normal order* N, i.e. relocate all a's right to $a^\dagger$'s. From this, one can rewrites the Hamiltonian (1.7.5) as

$$\begin{aligned} H|0\rangle &= \frac{1}{4}\int d^3\mathrm{k}\ [a(k), a^\dagger(k)]|0\rangle, \\ &= \delta^3(0)\int d^3\mathrm{k}\ \frac{\omega_k}{2}\delta^3(0), \end{aligned}$$

where the delta function is defined as

$$\delta^3(k-k') = \int \frac{d^3x}{(2\pi)^3} e^{-ix(k-k')}. \tag{1.8.3}$$

The normal ordered Hamiltonian is then given by

$$: H := \frac{1}{2}\int d^2\mathbf{k}\ a^\dagger(k)a(k). \tag{1.8.4}$$

$$: H : |a\rangle = 0. \tag{1.8.5}$$

From these results, an important completeness relation is given by

$$1 = |0\rangle\langle 0| + \int \frac{d^3k}{2\omega_k}|k\rangle\langle k| + \frac{1}{2}\int \frac{d^3k_1 d^3k_2}{2\omega_{k_2}\omega_{k_1}}|k_1k_2\rangle\langle k_1k_2|.... \tag{1.8.6}$$

1.9 Scattering and the S-Matrix

The theory we have been developed should be used to calculated the simplest process on field theories, a particle *scattering*. In such process there will an initial state, **IN**, which we say that is very far before the scattering moment, and a final state, **OUT**, taken a long time after the scattering moment. Once one assumes completeness of these both states and waiting long enough, any state will constitute a superposition of separable particles. Quantitatively one may say that

- All particles were separated: $\sum_\sigma |\alpha_{in}\rangle\langle\alpha_{in}| = 1$.
- All particles are eventually separated: $\sum_\sigma |\beta_{out}\rangle\langle\beta_{out}| = 1$.

The *S-matrix* is the amplitude for a system that was simple in the past and which will be simple in the future, and is defined by

$$S_{\alpha\beta} = \langle \beta_{out} | \alpha_{in} \rangle,$$

with the following proprieties,

1. **Completeness**: S is unitary.
2. **T-matrix**: another unitary matrix can be obtained by $S = 1 + iT$.
3. **Probabilistic interpretation**: the process $\beta \to \alpha$ has the probability $|S_{\alpha\beta}|^2$, and $\alpha \to \beta$, $|S_{\beta\alpha}|^2$.

Causal Green's Function

The *Green's functions* are connective solutions for equations of the type

$$(\Box_x + m^2) G(x - x') = \delta^4(x - x'),$$

which the solutions with sources are

$$(\Box_x + m^2)\phi = J.$$

In the scalar field Lagrangian such as (1.1.3), one can insert an additional quadratic term which represents the fields acting on itself

$$\mathcal{L} = |\partial\phi|^2 - m^2\phi^2 - \lambda(|\phi^2|)^2.$$

For such a problem, the solution of the equation of motion can be given by the causal Green's function

$$\begin{aligned} &(k^2 - m^2)\tilde{G}(k) = 1, \\ &\tilde{G}(\tilde{k}) = \int d^4x e^{-i(k-k')} G(x - x'), \\ &\tilde{G}(k) = \frac{1}{k^2 - m^2}. \end{aligned}$$

The Green's function has an important role on field theory, this connection propriety ultimately will be used to represent *transition amplitude* between states,

$$G(x_1, ..., x_N) = \langle 0 | T\phi(x)...\phi(x_n) | 0 \rangle. \tag{1.9.1}$$

The Reduction Theorem

The reduction theorem relates the Green's function for states, (1.9.1) to the S-Matrix

$$S(q_j,k_i) = \langle \{q_j\}_{out}|\{k_i\}_{in}\rangle,$$
$$S(q_j,k_i) = \frac{1}{\left(i(2\pi)^{\frac{3}{2}}R\right)^{n+m}}\left|\prod_{j=1}^{n}(q_j^2-m_j^2)\prod_{i=1}^{m}(k_i^2-m_i^2)\tilde{G}(q_i-k_i)\right|_{k_i^2=m_i^2,q_i^2=m_i^2},$$

where $R=(2\pi)^3|\langle 0|\phi(0)|\rho\rangle|^2$. In each isolated particle, $\tilde{G}$ has a separated pole and the residue is S. The construction is possible by considering a large time where the wave are isolated particles.

1.10 Path Integral and Feynman Diagrams

To start the study of *Path Integral*, we need to define the concept of *stationary phase*, which is the previous transition amplitude as the product of all possible trajectories for two classical coordinates q' and q'',

$$\begin{aligned} U(q'',t'',q',t') &= \langle q'',t''|q',t'\rangle, \\ &= \mathbf{lim}_{n\to\infty}\left(\frac{m}{2\pi i\delta_t\hbar}\right)^{\frac{n}{2}}\prod_{i=1}^{n}\int dq_i e^{\frac{1}{\hbar}\sum_j \delta_t L(q_k,\dot{q}_j)}. \end{aligned}$$

The stationary phase integral for the path integral, in the semi-integral approximation, is given by the classical variations. We can, define the *generating function* of $q(t)$ as

$$Z[J] = \int_{q_1}^{q_2}[dq]e^{\frac{1}{\hbar}[S-\int_{t'}^{t''}dtJ(t)q(t)]} \quad (1.10.1)$$

The transition for fields is made by the formal substitution $\int[dp][dq]\to\int[d\pi][d\phi]$, which can have the interpretation of transforming one harmonic oscillator to infinite number of harmonic oscillators, all labeled by their wave number k. The generating are then

$$\begin{aligned} W_{\delta a}[J(x^\mu)] &= \int[d\phi]e^{\frac{i}{\hbar}\int d^4x\frac{1}{2}[S_{\mathcal{L}},-\int d^4xJ(x)\phi(x)]}, \\ &= \int[d\phi]e^{\frac{i}{\hbar}\int d^4x\frac{1}{2}[(\partial^\mu\phi)^2-m^2\phi^2-J(x)\phi(x)]}, \end{aligned}$$

where the last term can be seen as a charge source.

Let us recall the procedure of *Wick's rotation* , which is the method of finding a solution to a problem in *Minkowski space* from a solution to a related problem in *Euclidean space*. The Wick's rotation consists in performing the transformation $t \to i\tau$, making it on $W[J]$ for the single degree of freedom. From $t[y] = e^{-i\theta\tau}$, $0 < \theta < \frac{\pi}{2}$, it is possible to write $W[J] = \lim_{\theta \to 0^+} W[J, \theta]$. From ((1.10.2)), it is possible to construct the free field Green's functions from the transition amplitude,

$$\langle 0|T(\phi(x)\phi(y)|0\rangle = i\Delta_f(x-y),$$

where the original equation of motion can be written as the

$$(\Box_x + m^2)\Delta_f(x-y) = -\delta^4(x-y),$$

where Euclidean delta function is

$$i\Delta_f(x) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{-ikx}}{k^2 - m^2 + i\epsilon}.$$

The Green's function, from i terms of the generating functional, equation (1.10.2), is then

$$G_N(x_1, ..., x_n) = \Big\langle 0 \Big| T\Big(\prod^n \phi(x_i)\Big)|0\Big\rangle, \tag{1.10.2}$$

$$= (i\hbar)^n \prod_{i=1}^n \frac{\delta}{\delta J(x_i)} W_{\delta_0}[J], \tag{1.10.3}$$

where

$$W_{\delta_0}[J] = e^{\frac{i}{2\hbar}\int d^4z d^4y J(z)\Delta_p(z-y)J(y)}. \tag{1.10.4}$$

Equation (1.10.4) gives the *Feynman Rules for scalars fields*, summarized as

1. Write all possible distinguished graphics for the process.
2. For each line, associate the propagator $\int \frac{d^4k}{(2\pi)^4} \frac{1}{k^2 - m^2 + i\epsilon}$.
3. For each vertex associate $-ig(2\pi)^4 g^4 \delta(\sum_i P_i - \sum_j k_j)$, where the delta function is over the sum of the momenta that come to the vertex.

Chapter 2

Fields with Spin

A more realistic kind of field theories are those that obey the *spin-statistic theorem*, where all particles have either integer spin, *bosons*, or half-integer spin, *fermions*, in units of the Planck constant $\hbar$.

2.1 Dirac Equation and Algebra

Pauli Matrices

The *Pauli matrices* are a set of 2×2 complex Hermitian and unitary matrices given by

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Together with the matrix identity, I, the Pauli matrices form an orthogonal basis. The sub-algebra of these matrices generate the real *Clifford algebra* of signature (3,0), and its proprieties are

- $\alpha_1^2 = \alpha_2^2 = \alpha_3^2$ =I= α_i^2,
- det (σ_i) = -1,
- Tr $(\sigma_i) = 0$,
- $[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k$,
- $\{\sigma_i, \sigma_j\} = 2\delta_{ij}I$

Dirac Equation

We now try to include the relativistic theory, represented by

$$E^2 = \vec{P}^2 + m^2, \tag{2.1.1}$$

In the *Schroedinger Equation*, we want to construct an equation that, unlike *Klein-Gordon* , is linear in ∂_t and is covariant (linear in ∇), with the general form

$$H\psi = (\vec{\alpha}.P + \beta m)\psi, \tag{2.1.2}$$

where $\vec{\alpha} = \sum_i \alpha_i$, with $i = 1, 2, 3$. Squaring 2.1.2 and comparing to 2.1.1 gives

$$H^2\psi = (P^2 + m^2)\psi, \tag{2.1.3}$$

with the following conditions:

- α_i, β all anti-commute with each other: $\{\alpha_i, \beta\} = 0$.
- $\alpha_i^2 = 1 = \beta^2$, so the anti-commutators are: $\{\alpha_i, \alpha_i\} = \alpha_i^2 + \alpha_i^2 = 2 = \{\beta, \beta\}$.

Clearly, ordinary numbers do not hold these proprieties, therefore we introduce 4×4 matrices operators (which are hermitian and traceless matrices of even dimensions, proprieties borrowed from the very first construction of the Pauli matrices, now extended in higher dimension), and consider the wave functions as column vector. The choice of the of these matrices are not unique. We will choose the *Dirac-Pauli representation*,

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \beta = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}.$$

The *Weyl or chiral representation* is given by the sets

$$\alpha_i^W = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \beta^W = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}.$$

Now back in 2.1.2. rewriting the operators H and P as $i\partial_t$ and $-i\partial_{x_i} = \nabla$, respectively (observe the metric $(+ - --)$), and multiplying β on the left of this equation,

$$i\beta\partial_t = (-i\beta\alpha\nabla + m)\psi, \tag{2.1.4}$$

gives the *covariant form* of the Dirac equation [1],

$$(i\gamma^\mu\partial_\mu - m)\psi = 0, \tag{2.1.5}$$

[1] Note that we were not worried about the covariant/contravariant form of our tensors because they were the same before. For now on we will work in the covariant form in the metric $(+ - --)$.

with the inclusion of the four *Dirac matrices* $\gamma^\mu = (\beta, \beta\alpha^i)$.

Clifford Algebra

From the constrains that we have found for β and α_i, we are going to derive the algebra of the Dirac matrices, the *Clifford algebra*. First, let us resume them in

$$\{\alpha^i, \alpha^j\} = 2\delta^{ij} \text{ and } \{\beta, \beta\} = 2 \text{ and } \{\alpha^i, \beta\} = 0.$$

It is clear that introducing a four-vector notation will summarize them. Let us from this derive the algebra of a four-vector representation $\gamma^\mu = (\beta, \beta\alpha^i)$, testing all possibilities of commutations. For $i \neq j$,

$$\begin{aligned}
\{\beta, \beta\alpha^j\} &= \beta\beta\alpha^i + \beta\alpha^i\beta \\
&= \beta^2\alpha^i - \alpha^i\beta^2 \\
&= 0. \\
\{\beta\alpha^i, \beta\alpha^j\} &= \beta\alpha^i\beta\alpha^j + \beta\alpha^j\beta\alpha^i \\
&= -\alpha^i\beta^2\alpha^j - \alpha^j\beta^2\alpha^i \\
&= \{\alpha^i, \alpha^j\}, \\
&= 0. \\
\{\beta, \beta\} &= \beta^2 + \beta^2, \\
&= 2.
\end{aligned}$$

Now, for $i = j$,

$$\begin{aligned}
\{\beta\alpha^i, \beta\alpha^i\} &= \beta\alpha^i\beta\alpha^i + \beta\alpha^i\beta\alpha^i, \\
&= 2\beta\alpha^i\beta\alpha^i, \\
&= -2\alpha^i\beta\beta\alpha^i, \\
&= -2.
\end{aligned}$$

Rewriting all in terms of γ-matrices,

$$\begin{aligned}
\beta &= \gamma^0, \\
\beta\alpha^1 &= \gamma^1, \\
\beta\alpha^2 &= \gamma^2, \\
\beta\alpha^3 &= \gamma^2,
\end{aligned}$$

we can see clearly that

$$\begin{aligned}
\{\gamma^0, \gamma^0\} &= 2, \\
\{\gamma^0, \gamma^i\} &= 0, \\
\{\gamma^i, \gamma^j\} &= -2\delta^{ij},
\end{aligned}$$

Resulting, finally, in the Clifford Algebra,

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{2.1.6}$$

where $g^{\mu\nu}$ is the element of the metric with the signature $(+,-,-,-)$.

We can prove 2.1.6 explicitly by actually substituting the Pauli matrices into $\gamma^\mu = (\beta, \beta\alpha_i)$. For example, for $\{\gamma^0, \gamma^2\}$,

$$\begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}\begin{pmatrix} 0 & \sigma^2 \\ -\sigma^2 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 & 0 & -i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix} = 0,$$

and for $\{\gamma^3, \gamma^3\}$,

$$\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^3 \\ -\sigma^3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} = -2.$$

2.2 Spinors

Recalling equation (1.4.1), one can write the general space-time transformations of fields as

$$\bar{\phi}_a(x) = S_{ab}(\Lambda)\phi_b(\Lambda^{-1}x - a),$$

which, near to the identity, can be writen as

$$S_{ab}(1 + \delta\lambda) = \delta_{ab} + \frac{1}{2}i\delta\lambda^{\mu\nu}(\Sigma_{\mu\nu})_{ab}.$$

The matrices $S(\Lambda)$ must obey the Poincar group proprieties $S(\Lambda_1)S(\Lambda_2) = S(\Lambda_1, \Lambda_2)$. The fields ϕ_b are tensors that transform according to the representation on $S(\Lambda)$. In quantum field theory the transformations are unitary, therefore the representation of the Poincare groups is given by the matrices

$$\Lambda^\mu_\nu = e^{i\omega.K}e^{-\theta.J} = S(\Lambda), \tag{2.2.1}$$

resulting on the Lorentz transformations of the fields,

$$U(a,\Lambda)\phi_a(x)U^{-1}(a,\Lambda) = S_{ab}(\Lambda^{-1})\phi_b(\Lambda x + a). \tag{2.2.2}$$

With the transformation matrices established, it is possible to construct the algebra for the Poincare group on field theory. The Poincare algebra, (1.4.3), gives the Lie algebra for the generators K and J on (2.2.1),

$$\begin{aligned} [J_a, J_b] &= i\epsilon_{abc}J_c, \\ [K_a, K_b] &= -i\epsilon_{abc}J_c, \\ [J_a, K_b] &= i\epsilon_{abc}K_c. \end{aligned} \tag{2.2.3}$$

Setting the normalization $T(j) = \frac{1}{2}$ such that tr $\{t_a, t_b\} = T(j)\delta_{ab}$, with j labeling an *irreducible representation*(irrep) of $SU(2)$, one can write the *Casimir operators* in terms of the generators,

$$\sum_{a=1}^{3}[t_a^{(j)}]^2 = j(j+1)I^j,$$

Let us consider the Pauli matrices, defined as

$$\sigma_0 = 1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \sigma_x = \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_y = \sigma_2 \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \sigma_3 = \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

A representation for the Lorentz group for 2×2 system in the gauge group $Sl(2, C)$, can be writen, and it is called *spinors*,

$$\begin{aligned} K_a &= -\frac{1}{2}i\sigma_a, \\ J_a &= \frac{1}{2}\sigma_a, \\ \tilde{K}_a &= \frac{1}{2}i\sigma_a^*, \\ \tilde{J}_a &= \frac{1}{2}\sigma_a^*. \end{aligned} \tag{2.2.4}$$

Substituting (2.2.4) back on (2.2.1), one has

$$\begin{aligned} h(\Lambda)_b^a &= (e^{\omega\cdot\frac{\sigma}{2}}e^{-i\theta\cdot\frac{\sigma}{2}})_b^a, \\ h^*(\Lambda)_{\dot{b}}^{\dot{a}} &= (e^{\omega\cdot\frac{\sigma^*}{2}}e^{-i\theta\cdot\frac{\sigma^*}{2}})_{\dot{b}}^{\dot{a}}, \end{aligned} \tag{2.2.5}$$

where $\dot{a}, a$ runs from $1, 2$, and in this representation, $h^{-1}(\Lambda) = h(\Lambda^{-1})$. Therefore, the spinor representation is double valued. For instance, for

rotation only one has $h^*(R) = \sigma_2 h(R)\sigma_2$, which is not necessary true when adding a *boost*.

The relation between $h(\Lambda)$ to Λ^μ_ν from (2.2.1) and (2.2.5) is conventionally given by taking the trace

$$\Lambda^\mu_\nu = \frac{1}{2}\,\mathrm{Tr}\,[\sigma^\mu h(\Lambda)\sigma_\nu h^\dagger(\Lambda)], \tag{2.2.6}$$

where $\sigma_\mu = (\sigma_0, \vec{\sigma})$. A rotational spinor $\eta^a = (\eta^1, \eta^2)$ is defined as a complex object that transform on the following way

$$\bar{\eta}^b = h(\Lambda)^b_a \eta^a, \tag{2.2.7}$$
$$\bar{\xi}^{\dot{b}} = h^*(\Lambda)^{\dot{b}}_{\dot{a}} \xi^{\dot{a}}. \tag{2.2.8}$$

They are the *Weyl* spinors and they give scalars by constructing

$$\eta_a = \epsilon_{ab}\eta^b,$$
$$\xi_{\dot{a}} = \epsilon_{\dot{a}\dot{b}}\xi^{\dot{b}},$$

where

$$\epsilon_{ab} = \epsilon_{\dot{a}\dot{b}} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and

$$\epsilon^{ab} = \epsilon^{\dot{a}b} = -\epsilon_{ab}.$$

These matrices are characteristic in the symplectic Lie groups and transform with the Pauli matrices as $\epsilon\sigma_i\epsilon^{-1} = -\sigma_i^T$. From this, one has

$$\begin{aligned}
\bar{\eta}_a &= [h^{-1}(\Lambda)]^c_a \eta_c \\
&= \eta_c [h(\Lambda)]^c_a, \\
\bar{\xi}_{\dot{a}} &= [h^{*-1}(\Lambda)^T]^{\dot{c}}_{\dot{a}} \xi_{\dot{c}} \\
&= \xi_{\dot{c}} [h^*(\Lambda)]^{\dot{c}}_{\dot{a}}.
\end{aligned}$$

Applying this last resulting in the previous calculation to find the scalars of the theory, one finds $\bar{\eta}_a\bar{\eta}^a = \eta_a\eta^a$, $(\xi^{\dot{a}})^*\eta_a$, and $(\eta_a)^*\eta_{\dot{a}}$.

2.3 Vectors

For a new field entity called *vector*, V^{μ}, the transformations can be defined as

$$\begin{aligned}(V)^{a\dot{b}} &= (V^{\mu}\sigma_{\mu})^{a\dot{b}} \\ &= (V_0\sigma_0 + V.\vec{\sigma})^{a\dot{b}},\end{aligned}$$

where $(\bar{V})^{a\dot{b}} = (\Lambda V)^{a\dot{b}}$. They transform as a tensor, on the same fashion as spinors,

$$\begin{aligned}(\bar{V})^{a\dot{b}} &= h(\Lambda)^{a}_{c} h^{*}(\Lambda)^{\dot{b}}_{\dot{d}} (V)^{c\dot{d}} \\ &= [h(\Lambda) V h^{\dagger}(\Lambda)]^{a\dot{b}}. \\ (\bar{V})_{a\dot{b}} &= \epsilon_{ac}\epsilon_{\dot{b}\dot{d}}(V)^{c\dot{d}} \\ &= V^{0}(\sigma_0)_{a\dot{b}} - V(\sigma^{T})_{a\dot{b}}.\end{aligned}$$

Partial derivatives are vectors on field theory, and they are defined obeying the following transformations proprieties

$$\begin{aligned}\partial^{\mu} &= (\partial^{0}, -\vec{\nabla}), \\ &= \frac{\partial}{\partial x_{\mu}}, \\ (\partial)^{a\dot{b}} &= (\partial^{0}\sigma_0 - \vec{\sigma}\nabla)^{a\dot{b}}, \\ (\partial)_{a\dot{b}} &= (\partial^{0}\sigma_0 + \vec{\sigma}^{T}\nabla)_{a\dot{b}}.\end{aligned}$$

Let us remember what we know from the the electromagnetic theory. One can define the fields $A^{\mu}(x)$, which are massless like the photon (and briefly become massive). The field strength is again defined as

$$F^{\mu\nu} = \partial^{\mu}A^{\nu} - \partial^{\nu}A^{\mu},$$

where

$$\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} = -\frac{1}{4}F^{2}, \tag{2.3.1}$$

is the Maxwell lagrangian, and we easily obtain the equations of motion,

$$\begin{aligned}&\partial_{\mu}(F^{\mu}_{\nu}) = 0, \\ &\partial_{\mu}\partial^{\mu}A_{\nu} - \partial_{\nu}(\partial^{\nu}A_{\mu}) = 0.\end{aligned}$$

The Lagrangian and the field strength are gauge invariants under $A'^{\mu} = A^{\mu} - \partial^{\mu}_{\alpha}(x)$ (invariance by a term that lives on the surface). One can use this fact to change the equation of motion, $\Box A'^{\mu} = 0$.

2.4 Majorana Spinors

One of first attempts of adding a mass term for the two-component spinors was done by the Majorana, called *Majorana spinors.* Weyl equations are massless and Dirac equations require that the spinors are indistinguishable, resulting on the Majorana Lagrangian

$$\mathcal{L}_M = (\eta^*)^{\dot{b}}(\partial)_{a\dot{b}}\eta^a + \frac{m}{2}(\eta_a\eta^a + \eta^*_{\dot{a}}\eta^{*\dot{a}}),$$

with the the Majorana equation of motion

$$(\partial)_{a\dot{b}}\eta^a + m(\eta^*)_{\dot{b}} = 0,$$

which is not consistent to the $U(1)$ symmetry, i.e. the phase symmetry.

2.5 Weyl Equation and Dirac Equation

The spinors we had previously defined have to satisfy the Klein-Gordon equation, (1.1.2),$(\Box + m^2)\eta^a(x) = 0$. When we make $m = 0$ we obtain the massless equation called *Weyl equation.* For the component spinors $\eta^a(x)$, one has

$$(\partial)_{a\dot{b}}\eta^a(x) = 0,$$
$$\text{multiplying by } (\partial)^{c\dot{b}} = (\sigma_0\partial^0 - \sigma.\nabla)^{c\dot{b}},$$
$$\text{one has } \Box u^a(x) = 0.$$

One can also derive the Weyl equation from the Lagrangian density $\mathcal{L} = u^{*\dot{b}}(\partial)_{a\dot{b}}u^a$, which is a Lorentz scalar. The generalization of the Weyl equation, when one includes mass, is called *Dirac equation*, to construct the Dirac equation, one defines one more spinor $\xi_{\dot{a}}$ such as in the table 2.5.

$\eta^a(x) \rightarrow$ Transforms as $h(\Lambda)$
$\xi_{\dot{a}}(x) \rightarrow$ Transforms as $[h^{-1}(\Lambda)]^\dagger$

Table 2.1: Spinor transformation in the Dirac/Weyl theory.

The two components are linked to the two spinors as

$$i(\partial)_{c\dot{b}}\eta^c(x) - m\xi_{\dot{b}}(x) = 0,$$
$$i(\partial)^{a\dot{d}}\xi_{\dot{d}}(x) - m\eta^d(x) = 0,$$

which can be written as a matrix in terms of σ_μ

$$\begin{pmatrix} -m\delta^{\dot d}_{\dot b} & i(\sigma_0\partial^0+\sigma\nabla)_{\dot b c} \\ i(\sigma_0\partial^0+\sigma\nabla)^{\dot a d} & -m\delta^c_a \end{pmatrix} \cdot \begin{pmatrix} \xi_{\dot d}(x) \\ \eta^c(x) \end{pmatrix} = 0,$$

The Dirac spinor representation in the 4×4 notation is

$$\psi(x)_D = \begin{pmatrix} \eta^a + \xi_{\dot a} \\ \eta^b - \xi^{\dot b} \end{pmatrix},$$

which transforms with the Dirac matrices

$$\gamma^i_D \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \gamma^0_D \begin{pmatrix} \sigma_0 & 0 \\ 0 & -\sigma_0 \end{pmatrix}.$$

The Weyl or chiral representation is given by the sets

$$\psi(x)_W = \begin{pmatrix} \xi_{\dot d}(x) \\ \eta^c(x) \end{pmatrix},$$

these 4×4 four-component spinors representation also transform by the Dirac matrices

$$\gamma^i_W \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \gamma^0_W \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}.$$

The Dirac equation can be written as

$$(i\partial_\mu\gamma^\mu - m)\psi(x) = 0, \tag{2.5.1}$$

where one defines

$$\begin{aligned} \bar\psi(x) &= \psi^\dagger\gamma_0, \\ &= (\psi^*)^T\gamma_0, \end{aligned}$$

and the Lorentz invariant is

$$\begin{aligned} (\bar\psi\psi) &= \bar\psi_\alpha\psi_\alpha \\ &= (\xi_{\dot a})^*\eta^a + (\eta^a)^*\xi_{\dot a}. \end{aligned}$$

The Dirac Lagrangian is

$$\mathcal{L} = \bar\psi(i\partial_\mu\gamma^\mu - m)\psi. \tag{2.5.2}$$

From unitary transformations we can write another representation,

$$\psi' = U\psi,$$
$$\gamma'^{\mu} = U\gamma U^{-1},$$

where

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} \sigma_0 & \sigma_0 \\ -\sigma_0 & \sigma_0 \end{pmatrix}.$$

The basic proprieties of the Dirac's matrices algebra, which is the *Clifford Algebra*, is given by the anticommutation of two matrices,

$$\{\gamma_\mu, \gamma_\nu\} = \gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu \tag{2.5.3}$$
$$= 2g_{\mu\nu}(I_{4\times 4}), \tag{2.5.4}$$

where, for the 4×4 representation, one has the possible representations in the table 2.2.

Number of γ	0	1	2	3	4
Element	I	γ^μ	$\sigma^{\mu\nu}$	$\gamma^\mu\gamma^5$	γ^5
Quantity	1	4	6	4	1

Table 2.2: The Clifford matrices for the 4×4 representation.

2.6 Lorentz Transformations

As we have seen on (1.4.2) and (2.2.1), in general, near to the identity, one can write the generators of the group, such as the Lorentz group, in the form of the expansion

$$S_{ab}(1 + \delta\lambda) = I + \frac{1}{2} i\delta\lambda(\Sigma_{\mu\nu})_{ab}. \tag{2.6.1}$$

In the case of Dirac's equation, conventionally one has

$$(\Sigma_{\mu\nu})_{\alpha\beta} = -\frac{1}{2}(\sigma_{\mu\nu})_{\alpha\beta}, \tag{2.6.2}$$

where the new tensor is defined as

$$\sigma_{\mu\nu} = \frac{i}{2}[\gamma_\mu, \gamma_\nu],$$

with the following commutation relation to γ_ρ,

$$[\sigma_{\mu\nu}, \gamma_\rho] = 2i(g_{\nu\rho}\gamma_\mu - g_{\mu\rho}\gamma_\nu).$$

Hence, it is possible to rewrite equation (2.6.1) as

$$S(1+\delta\lambda) = I - \frac{1}{4}i\delta\lambda(\sigma_{\mu\nu}), \tag{2.6.3}$$

where the finite transformation is as an **arbitrary boost** plus an **arbitrary rotation**,

$$S(\Lambda)_{\alpha\beta} = [e^{\frac{1}{2}\omega_i\sigma_{0i}}e^{-\frac{1}{4}\theta_i\epsilon_{ijk}\sigma_{jk}}]_{\alpha\beta}. \tag{2.6.4}$$

Finally, we have the following important relations with the Dirac's matrices,

$$\begin{aligned} S^\dagger(\Lambda)\gamma_0 &= \gamma_0 S^{-1}(\Lambda), \\ S^{-1}(\Lambda)\gamma^\mu S(\Lambda) &= \Lambda^\mu_\nu\gamma^\mu, \\ \Lambda^\mu_\alpha\bar{\partial}^\alpha &= \partial^\mu. \end{aligned}$$

2.7 Symmetries of the Dirac Lagrangian

The Dirac Lagrangian, given by (2.5.2), can be written in a more compact way in the slash-notation, where $\partial\!\!\!/ = \gamma_\mu\partial^\mu$,

$$\mathcal{L}_D = \bar{\psi}(i\partial\!\!\!/ - m)\psi,$$

This Lagrangian is invariant under the gauge groups $SU(N)$ and $U(1)$.

Noether's Theorem

In the same fashion as the theory for scalar fields, one can define the conserved current, (2.7.1), for spinors

$$J^\alpha_a = -\left(\frac{\partial(\delta x^\alpha)}{\partial\beta^a}\right)\mathcal{L} - \sum_i \frac{\partial\mathcal{L}}{\partial(\partial_a\phi_i)}\frac{\partial\delta_*\phi_i}{\partial\beta_a},$$

where a is the parameter of transformation, α the vector index, the Poincare terms are $\beta_a\delta x^\mu, \delta\lambda^{\mu\nu}$, and the last term is the variation at a point. Looking to the form of the change of a field on a point on the Dirac's theory, one has

$$\begin{aligned}\delta x^{\alpha} &= \delta a^{\alpha} + \frac{1}{2}\delta\lambda^{\alpha}_{\mu}x^{\mu},\\ \delta_{*}\phi_i(x) &= -(\delta^{\nu}_{a}g^{\mu}_{\nu} - \delta\lambda^{\nu\mu}\delta x_{\nu})\frac{\partial\phi_i(x)}{\partial x^{\mu}} - \frac{i}{2}(\Sigma_{\mu\nu})_{ij}\delta\lambda^{\mu\nu}\phi_j(x),\end{aligned}$$

where, from (2.6.2) we know that $(\Sigma_{\mu\nu})_{\alpha\beta} = \frac{1}{2}(\sigma_{\mu\nu})_{\alpha\beta}$, and the last part of this equation is a new term different from the scalar theory. The Noether's current is

$$\begin{aligned}H_D &= \int d^3x\ \mathrm{T}^{00}_{D},\\ &= \int d^3x(-\mathcal{L} + i\psi^{\dagger}\partial_0\psi),\\ &= \int d^3x\bar{\psi}(-i\gamma\nabla + m)\psi.\end{aligned}$$

The conjugate momenta in the Dirac's theory is

$$\Pi_{\alpha} = \frac{\partial\mathcal{L}}{\partial\dot{\psi}_{\alpha}} = i\psi^{\dagger}_{\alpha}, \tag{2.7.1}$$

and the momentum-energy tensor is

$$\begin{aligned}\delta a^{\mu} &: \frac{\partial}{\partial(\delta a^{\mu})} \rightarrow T^{\mu\alpha}.\\ \delta\lambda^{\mu\nu} &: \frac{\partial}{\partial(\delta\lambda^{\mu\nu})} \rightarrow M^{\mu\nu\alpha}.\end{aligned}$$

The quantities $P_{\nu} = \int d^3xT_{0\nu}$ and $J_{\nu\lambda} = \int d^3xM_{0\nu\lambda}$ have the usual interpretation as the *total momentum* and the *angular momentum tensor*,

$$\begin{aligned}J^{\nu\lambda} &= \int d^3xM^{\nu\lambda_0},\\ &= \int d^3x(x_{\nu}T_{0\lambda} - x_{\lambda}T_{0\nu}) + i\int d^3x\Pi_i(\Sigma_{\nu\lambda})_{ij}\phi_j,\end{aligned}$$

where the second term is the *spin/intrinsic angular momentum*, and Π_i, ϕ_j are the fields ψ's. The new contribution describes the intrinsic spin and a measure of it is give by the *Pauli-Lobanski vector*, :

$$\begin{aligned}W_{\mu} &= -\frac{1}{2}\epsilon_{\mu\nu\lambda\sigma}J^{\nu\lambda}P^{\sigma},\\ &= -\frac{1}{2}i\epsilon_{\mu\nu\lambda\sigma}\int d^3x\Pi_i(\Sigma_{\nu\lambda})_{ij}\phi_jP^{\sigma},\end{aligned}$$

where $W^2 = W^\mu W_\mu$ and $P^2 = P^\mu P_\mu$ are Casimir operators of the theory . P^2 commutes with W^2, eigenvalues of W^μ. In the rest frame, the momentum vector reduces to a 3-vector, while in the other frames $W^\mu P_\mu = 0$.

Global Symmetries

The group of the *global symmetries* is $U(N) = U(1) \times SU(N)$. We can construct such symmetry by replicating fields with same mass

$$\mathcal{L} = \sum_{i-1}^{N} (\bar{\psi}_i)_\alpha (i\partial\!\!\!/ - m)_{\alpha\beta} (\psi_i)_\beta.$$

The Lagrangian is invariant by the transformation of $(\psi_i)_\alpha$,$i = 1, ..., N$, under $e^{i\theta}$ $(U(1))$ and $e^{i\sum_{a=1}^{N^2-1} \beta_a T_a}$ $(SU(N))$. This form-invariance produces conserved currents and charges, from the Noether's theorem,

$$\begin{aligned} J^\mu &= \frac{\partial \mathcal{L}}{\partial(\partial_\alpha \psi_i)} \delta_* \psi_i, \\ \text{U(1)} &\rightarrow J^\alpha = \bar{\psi}\gamma^\mu \psi, \\ \text{SU(N)} &\rightarrow J^\alpha_a = \bar{\psi}\gamma^\mu T_a \psi, \\ Q_a &= \int d^3x J^0_a. \end{aligned}$$

Parity

If $\psi(x)$ solves the Dirac equation, so does $\gamma_0 \psi(x) = \psi(x_0, -x)$. In the Weyl notation, the γ_0 exchanges the position of the dotted by the undotted and for this reason the Weyl Lagrangian] density is not form invariant under parity transformation.

A fifth gamma matrix is defined as

$$\gamma_5 = i\gamma_0\gamma_1\gamma_2\gamma_3 = \frac{i}{4!}\epsilon_{\mu\nu\lambda\sigma}\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma. \tag{2.7.2}$$

$$\{\gamma_5, \gamma_\mu\} = 0.$$

$$\gamma_5^W = \begin{pmatrix} -\sigma_0 & 0 \\ 0 & \sigma_0 \end{pmatrix}$$

$$\gamma_5^D = \begin{pmatrix} 0 & \sigma_0 \\ \sigma_0 & 0 \end{pmatrix}$$

From this new matrix γ_5, one can define the *projective operators* , that project out states in the Weyl representation. The so-called *chiral representation* is given by

$$\frac{1}{2}(1-\gamma_5)\psi = \begin{pmatrix} \xi_{\dot{d}} \\ 0 \end{pmatrix} \rightarrow Left - handed.$$

$$\frac{1}{2}(1+\gamma_5)\psi = \begin{pmatrix} 0 \\ \eta_c \end{pmatrix} \rightarrow Right - handed.$$

2.8 Gauge Invariance

One of the most important *gauge fixing* is the *Lorentz gauge* given by

$$\partial^\mu A_\mu = 0.$$

It is possible to impose it from the beginning by modifying the Maxwell Lagrangian itself

$$\mathcal{L}_M(A,\lambda) = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\lambda(\partial_\mu A^\mu),$$

This is no longer gauge invariant. Acting ∂^ν on $\Box A_\nu - (1-\lambda)\partial_\nu(\partial A) = 0$ results in $\lambda\Box(\partial A) = 0$.

Proca Field

The *Proca field* is the generalization of a massive vector field. The Lagrangian is given by the Maxwell Lagrangian plus a mass term on the vector fields

$$\mathcal{L}_P = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} + m^2 A_\mu A^\mu, \tag{2.8.1}$$

with equation of motion

$$\partial_\mu\partial^\mu A_\nu - \partial_\nu(\partial^\mu A_\mu) + m^2 A_\nu = 0.$$

If one acts ∂_ν on this equation, it is possible to get the Lorentz condition back, which is Lorentz invariant, such as the Klein-Gordon equation. Since it is not possible to remove the third number of degree of freedom, the Proca

Lagrangian is not gauge invariant. A solution for this massive vector field can be written as

$$A^{\mu}(k,x) = a^{\mu}.e^{-ikx},$$
$$\text{with } k^2 = a^2.$$

A massive vector has two transverse and one longitudinal *degrees of freedom.* For a massless vector, the remotion of the third degree of freedom comes from the polarization, $\epsilon_{long} = (|k|, 0, 0, \omega_k)$, where $k_{\mu}.\epsilon^{\mu}_{long} = 0$

Local Gauge Invariance

Let us perform the transformation $A'(x) = A(x) - \partial\alpha(x)$ in the Maxwell Lagrangian, $\mathcal{L}_M\dot{}$This is described as a local invariance. Combining it together with the local generalization of the global $U(N)$ and $SU(N)$ transformations of the scalar and the Dirac fields, it requires a vector field, i.e. the local gauge invariance requires a new interaction term in the Lagrangian. Examples of coupling are

- Vector and Dirac spinor, $\bar{A}_{\mu}\gamma^{\mu}\psi$,
- Scalars, $\phi(\bar{\psi}\psi), \pi(\bar{\psi}\gamma^5\psi)$,
- Vectors, $(A_{\mu}A^{\mu})$,
- Four-Dirac spinors, $[\bar{\psi}\gamma^{\mu}(a + b\gamma^5)\psi][\bar{\psi}\gamma^{\mu}(a - b\gamma^5)\psi]$.

Example: QED Coupling

The free Dirac equation has the $U(1)$ invariance,

$$\psi' = e^{ie\alpha(x)}\psi, \quad (2.8.2)$$
$$\bar{\psi}' = \bar{\psi}e^{-e\alpha(x)}, \quad (2.8.3)$$

which means that the Lagrangian is invariant to a multiplication of a phase. Making this phase the electric charge of the electromagnetism theory, a coupling of a vector field to the Dirac Lagrangian will not become form invariant

$$\mathcal{L}_{QED}[\bar{\psi}, \psi'] = \mathcal{L}_D[\bar{\psi}, \psi] - \epsilon\bar{\psi}\partial\!\!\!/\psi.$$

However it can be made invariant by adding a *minimal coupling* of the Maxwell field to the Dirac current. Using the notation of covariant derivative, the minimal coupling is

$$D_{\mu}[A] = \partial_{\mu} + ieA_{\mu}(x),$$

which allow us to write the QED Lagrangian

$$\begin{aligned}\mathcal{L}_{QED}(\bar{\psi},\psi,A) &= \bar{\psi}\Big(i\not{D}[A]-m\Big)\psi-\frac{1}{4}F^2,\\ &= \bar{\psi}\Big(i(\partial_\mu+ieA_\mu)\gamma^\mu-m\Big)\psi-\frac{1}{4}F^2.\end{aligned}$$

The quantum electrodynamic theory is then represented by the Lagrangian

$$\mathcal{L}_{QED}=\mathcal{L}_D+(i\epsilon A^\mu\bar{\psi}\gamma_\mu\psi). \tag{2.8.4}$$

Method of Wigner

Wigner developed a method by which representations of the Poincare group can be constructed explicitly. We start with a eigenstate $|p,\lambda\rangle$ of P^2 where

$$P^\mu|p,\lambda\rangle=p^\mu|p,\lambda\rangle.$$

The operators versions of P^μ and J^μ generate unitary representations of Poincare group, and one can treat P^μ and J^μ as hermitian operators on the space of states.

Fixing p^2 degenerates the basis states for all values of P^μ, and denoting the special fixed vector by q^μ, we have $q^2=m^2$. Any vector P^μ with $p^2=m^2$ can be derived by acting on q^μ with the appropriate Lorentz transformation $P^\mu=\Lambda(p,q)^\mu_\nu q^\nu$. The transformations is not unique, it can be modified to $\hat{\Lambda}(p,q)=\Lambda(p,q)l$. The set of all transformations l that satisfies $lq=q$ is a group, the *Wigner little group*. The set of states with momentum q^μ which transforms according to irreps of the little group are

$$U(l)|q^\mu,\lambda\rangle=\sum_{\sigma\lambda}(l)|q^\mu,\sigma\rangle.$$

To find the representations of l, for example if $m^2>0$, one chooses $q^\mu=(m,0)$ (vector at rest). The l is the rotation group and the states $|q^\mu,\sigma\rangle$ are most conveniently chosen with σ as the projector of the spin along the fixed axis.

The ambiguity of the choice of Λ that takes q^μ into p^μ can be removed by making a specific choice of Wigner boost, $\Lambda_0(p,q)$, for each p^μ. A example is making $p^\mu\neq q^\mu$ and getting $|p^\mu,\lambda\rangle=U(\Lambda_0(p,q)|q^\mu,\lambda\rangle$.

The case $k = 0, p^\mu = (m, 0)$. We want the solution for the Dirac equation

$$(i\partial\!\!\!/ - m)_{\alpha\beta}\xi^{\pm}_{\beta}(q)e^{\mp iq_0x_0} = 0,$$
$$\text{which is } (i(\mp i)\gamma^0 q^0 - m)_{\alpha\beta}\xi^{\pm}_{\beta}(q) = 0.$$

In the matrix representation:

$$\begin{pmatrix} (\pm q_0 - m)\sigma_0 & 0 \\ 0 & -(\pm q_0 - m)\sigma_0 \end{pmatrix} \xi^{\pm} = 0.$$

We find four independent solutions,

$$u_\alpha(q, \pm\frac{1}{2})e^{-iqx}, v_\alpha(q, \pm\frac{1}{2})e^{iqx}.$$

For instance, the two solutions are the positive energy and the two last are the negatives,

$$u_\alpha(q, +\frac{1}{2}) = C \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix},$$

where C is the normalization constant, here chosen to be $C = \sqrt{2m}$.

The case $k \neq 0, q \neq p$. We want the solution

$$(q\!\!\!/ - m)\xi^{\pm}_{\beta}(\lambda) = 0,$$
$$S(\Lambda(p,q))q\!\!\!/ S^{-1}\Lambda(p,q) = p\!\!\!/,$$

and the infinite solutions

$$(p\!\!\!/ - m)[S(\Lambda(p,q)\xi^{\pm}(\lambda)] = 0.$$

We need a unitary transformations, however, in general, spinors transformations are non-unitary $S(\Lambda)$, making it necessary to have

$$U(\Lambda)|k^{\pm}, \lambda\rangle = \sum_{\sigma} D_{\lambda\sigma}(\Lambda)|\Lambda k^{\pm}, \sigma\rangle,$$
$$\text{where } D^{\dagger}D = 1.$$

This equation requires then

$$U(\Lambda)b^{\dagger}_{\lambda}(k)U(\Lambda^{-1}) = \sum_{\sigma} D_{\lambda\sigma}(\Lambda)b^{\dagger}_{\sigma}(\Lambda k),$$
$$\text{consistent to } U\psi(x)U^{-1} = S(\Lambda^{-1})\psi(\Lambda x).$$

Hence we construct a set of spinor solutions that transforms arbitrary by Dirac. In this case, the conservation of probability requires

$$U(\Lambda)b^{\dagger}U^{-1}(\Lambda) = \sum_{\sigma} D_{\lambda\sigma}(\Lambda)b^{\dagger}_{\sigma}(\Lambda k).$$

Now let us look to $S_{\alpha\beta}(\Lambda_d^{-1}(k,\lambda)$, where we use the Wigner boost to define

$$\begin{aligned}
&u_{\alpha}(k,\lambda) = S_{\alpha\beta}(\Lambda_w(k,q))u_{\beta}(q,\lambda),\\
&S(\Lambda^{-1})u_{\alpha}(k,\lambda) = S(\Lambda^{-1})S_{\alpha\beta}(\Lambda_w(k,q))u_{\beta}(q,\lambda).
\end{aligned}$$

However $S(\Lambda_w(q,k'))S(\Lambda^{-1})S(\Lambda_w(k,q)) = S(\Lambda_w(q,k')\Lambda^{-1}\Lambda_w(k,q) = S(l_{k',k,q})$ is a rotation! This results that

$$\begin{aligned}
S_{\alpha\beta}(\Lambda^{-1})u_{\beta}(k,\lambda) &= \sum_{\sigma} D_{\lambda\sigma}(l_{k',k,q})u_{\alpha}(k',\sigma),\\
&= \int d\tilde{k}\sum_{\lambda\sigma} b_{\lambda}(\Lambda k')D_{\lambda\sigma}(l_{k',k,q})u(k',\sigma)e^{-ikx},
\end{aligned}$$

with the equation insured by

$$U(\Lambda)b_{\sigma}(k)U^{-1}(\Lambda) = \sum_{\lambda} b_{\lambda}(\Lambda k)D_{\lambda\sigma}(l_{k',k,q}).$$

Finally, in resume, the two steps for finding these kind of solutions are

1. Define ((2.8.5)).
2. Label solutions by rotation (Wigner's little group), which are induced solutions.

The Wigner's method works for any field with $q^2 = m^2$. For $q^2 = 0$, we should choose different reference momentum q^{μ}, and it is possible do it for*tachyons*.

From here, it is easy to construct basis of solutions to Dirac equation: using the spin (helicity) basis, $\Lambda_w(k,q)$ is a pure boost inp-direction with

$$\begin{aligned}
&\Lambda_0(p,q) = e^{i\omega p.k},\\
&\omega = \tanh^{-1}\Big(\frac{|p|}{\sqrt{p^2+m^2}}\Big),\\
&p^{\mu} = \Lambda_0(p,q)^{\mu}_{\nu}q^{\nu},\\
&q^{\mu} = (m,\vec{0})^{\alpha\beta}\Lambda_0(p,q) = e^{i\omega p.k}.
\end{aligned}$$

To evaluate the solutions, we denote $\omega = u, v$ the spin basis

$$\omega(p,\lambda) = S(\Lambda_0(p,q))\omega(q,\lambda),$$
$$\text{where } S_{\alpha\beta}(\Lambda_0(p,q)) = e^{-\frac{1}{4}\omega\hat{p}_i[\gamma_0,\gamma_i]_{\alpha\beta}}$$

Evaluating this in the Dirac algebra gives

$$S(\Lambda_0(p,q)) = e^{\frac{1}{2}\omega_p.\alpha},$$
$$S(\Lambda_0(p,q)) = I \cosh\frac{\omega}{2} + \hat{\phi}\alpha \sinh\frac{\omega}{2},$$

and with a little algebra one gets the Dirac representations.

Sumating it up, the basis u and v can be written as:

$$u(p,\lambda) = \frac{1}{\sqrt{2m(p_0+m)}}(\not{p}+m)u(q,\lambda),$$
$$v(p,\lambda) = \frac{1}{\sqrt{2m(p_0+m)}}(-\not{p}+m)v(q,\lambda),$$

and it is possible to make also use of

$$S^\dagger(\lambda)\gamma_0 = \gamma_0 S^{-1}(\Lambda),$$
$$\bar{u}(p,\lambda) = \frac{1}{\sqrt{2m(p_0+m)}}(-\not{p}+m)\bar{u}(q,\lambda),$$
$$\bar{v}(p,\lambda) = \frac{1}{\sqrt{2m(p_0+m)}}(\not{p}+m)\bar{v}(q,\lambda).$$

The final proprieties are

1. $(\not{p}-m)u = \bar{u}(\not{p}-m) = 0.$
2. $(\not{p}+m)v = \bar{v}(\not{p}+m) = 0.$
3. The normalization of the scalars are $\bar{u}(p,\lambda)u(p,\lambda') = |C|^2\delta_{\lambda\lambda'}$, $\bar{v}(p,\lambda)v(p,\lambda') = -|C|^2\delta_{\lambda\lambda'}$.
4. The *projector* for u are $\sum_{\lambda=\pm\frac{1}{2}} u_\lambda(p,\lambda)\bar{u}_\beta(p,\lambda) = \frac{|C|^2}{2m}(\not{p}+m)_{\alpha\beta}$, and for v are $\sum_{\lambda=\pm\frac{1}{2}} v_\lambda(p,\lambda)\bar{v}_\beta(p,\lambda) = \frac{|C|^2}{2m}(\not{p}-m)_{\alpha\beta}$.

2.9 Canonical Quantization

Making use of the spin representations, which are unitary representations of the Poincare group, one can organizes fields in terms of solution to the classical equation of motion,

1. Expanding to equation of motion.
2. Quantizating the coefficients.
3. Constructing the Hamiltonian H_0.
4. Finding the x_0-dependence of coefficients for free field.
5. Constructing the conserved charges.

From the classical transformations of the Dirac spinors, $S_{\alpha\beta}(\Lambda)\psi_\beta(x) = \psi'_\alpha(\Lambda x)$ of, the expansion of the field has the general form (in the same fashion as we derived for scalar fields),

$$\psi_\alpha(x) = \sum_\lambda \int \frac{d^3k}{(2\pi)^{\frac{3}{2}} 2\omega_k} [b_\lambda(k) u_\alpha(k,\lambda) e^{-ikx} + d^\dagger_\lambda(k) v_\alpha(k,\lambda) e^{i\bar{k}x}], \tag{2.9.1}$$

where the Dirac spinors, u_α, v_α, are some bases of solution, and the transformations of the fields are given by

$$U(\Lambda)\psi_\alpha U^{-1}(\Lambda) = S_{\alpha\beta}(\Lambda^{-1})\psi_\beta(\Lambda x).$$

Quantization

Let us know understand (2.9.1). Due the fact that the Dirac Lagrangian is linear in derivatives, the canonical momentum of the field is simply its conjugate, as in (2.7.1),

$$\Pi = i\psi^\alpha,$$

making the construction of the fields straightforward

$$\psi_\alpha(x, x_0) = \int d\tilde{k} \{ b_\lambda(k, x_0) u_\alpha(k,\lambda) e^{ikx} + d_\lambda(k, x_0) v^*_\alpha(k,\lambda) e^{-ikx} \},$$

$$\Pi_\alpha(x, x_0) = \int d\tilde{k} \{ b^\dagger_\lambda(k, x_0) u^\dagger_\alpha(k,\lambda) e^{-ikx} + d_\lambda(k, x_0) v^\dagger_\alpha(k,\lambda) e^{ikx} \}.$$

The Hamiltonian of the theory is then

$$\begin{aligned} H_0 &= \int d^3x\{i\psi^\dagger\partial^0\psi - \psi^\dagger\gamma^0(i\gamma^0\partial^0 + (i\gamma\nabla - m))\psi\}, \\ &= \int d^3x\bar{\psi}\{-i\gamma\nabla + \gamma^0 m\}\psi, \\ &= \int d^3k\frac{1}{2}\sum_\lambda\{b^\dagger_\lambda(k,x_0)b_\lambda(k,x_0) - d_\lambda(k,x_0)d^\dagger_\lambda(k,x_0)\}. \end{aligned}$$

The Hamiltonian normal-ordered is

$$: H_0 := \frac{1}{2}\sum_\lambda\int d^3k\{b^\dagger_\lambda(k,x_0)b_\lambda(k,x_0) + d^\dagger_\lambda(k,x_0)d_\lambda(k,x_0)\}.$$

The canonical equal time anti-commutation relations are

$$\{b^\dagger_\lambda(k,x_0), b_{\lambda'}(k',x'_0)\} = 2\omega_k\delta_{\lambda\lambda'}\delta^3(k-k'),$$
$$\{d^\dagger_\lambda(k,x_0), d_{\lambda'}(k',x'_0)\} = 2\omega_k\delta_{\lambda\lambda'}\delta^3(k-k'),$$

$$[b_\lambda(k,x_0), : H_0 :] = ib_\lambda(k,x_0).$$

The states can be constructed by acting the raising/lowering operators

$$d(k,\lambda)|0\rangle = b(k,\lambda)|0\rangle = 0,$$
$$\langle 0|d^\dagger = \langle 0|b^\dagger = 0.$$

and

$$\delta_{\{i,j\}}\prod_i b^\dagger_{\lambda i}(k'_i)\prod_k d^\dagger_{\lambda j}(kj)|0\rangle = |\{k'_i,\lambda_i\},\{k_j,\lambda_j\}\rangle,$$

where the delta has $\pm$ sign depending on the order of b and d. In the right hand side, the first term is the particle and the second is the antiparticle. The terms $b^\dagger$ creates particles and $d^\dagger$ creates antiparticles, and the difference to the scalars is that now one has a antisymmetric wave function.

Momentum and Spin

In terms of creation/annihilation operators, the momentum is just like the scalar,

$$\begin{aligned} P_i &= \int d^3x T_{0i} \\ &= \sum_\lambda \frac{d^3k}{2\omega_k}k[b^\dagger_\lambda(k)b_\lambda(k) + d^\dagger_\lambda(k)d_\lambda(k)]. \end{aligned}$$

On another hand, the angular momentum need to be redefined,

$$: J_{ij} := \int d^3x[x_i T_{0j} - x_j T_{0i} + i\Pi\sigma_{ij}\psi],$$

where its expected value is

$$\langle q, \lambda| : J_{ij} : |0, \lambda\rangle = \delta^3(q)u^\dagger(q, \lambda')\frac{1}{2}\sigma_{ij}u(q, \lambda).$$

Up to a normalization, the third component of the angular momentum in each state is just the expected value of J_{12}, which is $\sigma_{12}u = \lambda u, \sigma_{12}v = -\lambda u$. The spin-statistics theorem says that spin half-integers are fermions and spin integers are bosons.

The Lagrangian is Lorentz invariant and local gauge invariant.The charges $U(1)$ are then given by

$$\begin{aligned} : Q : &= \sum_\lambda \int \frac{d^3k}{2\omega_k}[b^\dagger_\lambda(k)b_\lambda(k) - d^\dagger_\lambda(k)d_\lambda(k)] \\ &= \int d^3x\bar{\psi}\gamma^0\psi, \end{aligned}$$

which is exactly - Q, the electric charge.

Proca Fields Revisited

Using the Wigner method, it is possible to solve the Proca equation of motion, (2.8.1), with a unitary matrix Λ although A^μ transforms under Λ, i.e, it is not unitary under this matrix. For that, we make use of the induced representations, starting with a wave vector $q^\mu = (m, \vec{0})$. The basic equations of Klein-Gordon, (1.1.2), for each A^μ are

$$(\Box + m^2)A^\mu(x) = 0,$$

where the basic solution for Proca must have $A_0(q) = 0$,

$$\epsilon^\mu(q, \lambda)e^{\pm imx_0} = A^\mu(q, \lambda),$$

with $\lambda = \pm 1, 0$ and

$$\epsilon(q, \pm 1) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ 1 \\ \pm 2 \\ 0 \end{pmatrix}, \epsilon(q, 0) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The boosted solutions are found in the same way as before, now with $S(\Lambda)_A = \Lambda$, for either massive or massless field,

$$A^\mu(x) = \int d\tilde{k} \sum_{\lambda=0,\pm 1} [a_\lambda(k)\epsilon^\mu(k,\lambda)e^{-ikx} + a^\dagger_\lambda(k)\epsilon^{*\mu}(k,\lambda)e^{ikx}],$$

In the interaction case, x_0 depends in $a, a^\dagger$. The projection is given by

$$\sum_\lambda \epsilon^{*\mu}(k,\lambda)\epsilon^\nu(k,\lambda) = -g^{\mu\nu} + \frac{k^\mu k^\nu}{m^2}.$$

Massless Field

To turn the solution of the last section to massless free-Lagrangian, we can derive it in the *light-cone coordinates formalism* $V^\mu = (V^0, V)$, with

$$\begin{aligned} V^\pm &= \frac{1}{2}(V^0 \pm V^3), \\ V^2 &= V_0^2 - V^2, \\ &= 2V^+V^- - V_{-T}^2, \\ V_{-T} &= (V_1, V_2), \\ d^4V &= dV^0 d^3V, \\ &= dV^+ dV^- d^2V_{-T}, \end{aligned}$$

where $(\gamma^\pm)^2 = \gamma$ and $\not{p}\not{p} = p^2$. Hence, for two vectors:

$$\begin{aligned} a.b &= a_0 b_0 - ab, \\ &= a^+b^- + a^-b^+ - ab, \\ \not{q} &= q^\mu \gamma_\mu, \\ &= q^+\gamma_+, \\ &= q^+\gamma^-. \end{aligned}$$

The induced representations for massless fields are the solution of the wave equation $\Box\phi = 0$, which are e^{ikx}, with $k^2 = 0$.

Starting with the reference vector analog to $q^\mu = (m, 0)$, the standard choice is $q^\mu = q^+\delta_{\mu+} = \frac{1}{2}(q^0 + q^3)$. Then it is necessary to find the analog for the rotation group, the Wigner's little group. The analogs of the generator of rotation for $(m, 0)$ are $m_{12}, \pi_1 = \sqrt{2}m_{1+}, \pi_2 = \sqrt{2}m_{2-}$. The Lie algebra is then defined by

$$[\pi^1, \pi^2] = 0,$$

$$[m_{12}, \pi_1] = i\pi_2,$$
$$[m_{12}, \pi_2] = -i\pi_1.$$

All the representations are one-dimensional or zero-dimensional. For a massless Dirac $\not{q}W(q, \lambda) = 0$, the equation is very simples,

$$(\not{q} = q^\mu \gamma_\mu = q^+ \gamma_+),$$

resulting in $\not{q}^+ \gamma^- W(q, \lambda) = 0$. The solutions are then

$$(\gamma^-)_{\alpha\beta} W_\beta = \sqrt{2}(\delta_{23} W_1 + \delta_{12} W_4),$$

$$(\gamma^-)_{\alpha\beta} \begin{pmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{pmatrix} = \sqrt{2} \begin{pmatrix} 0 \\ W_4 \\ W_1 \\ 0 \end{pmatrix} \to W_1 = W_4 = 0$$

Hence, one has positive and negatives solutions, labeled as u, v:

$$u(q, \frac{1}{2}) = v(1, -\frac{1}{2}) = 2^{\frac{1}{2}} \sqrt{q^+} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix},$$

$$u(q, -\frac{1}{2}) = v(1, \frac{1}{2}) = 2^{\frac{1}{2}} \sqrt{q^+} \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}.$$

The proprieties of the solutions are

- Orthonormality: $\bar{u}(q, \lambda) u(q, \sigma) = \delta_{\lambda\sigma}$.
- Projections: $\bar{u}(q, \frac{1}{2})_\alpha u(q, -\frac{1}{2})_\beta = \bar{v}(q, -\frac{1}{2})_\alpha v(q, \frac{1}{2})_\beta = \frac{1}{2}[(1 + \gamma_5)\not{q}]_{\alpha\beta}$ and $\bar{u}(q, -\frac{1}{2})_\alpha u(q, -\frac{1}{2})_\beta = \bar{v}(q, \frac{1}{2})_\alpha v(q, \frac{1}{2})_\beta = \frac{1}{2}[(1 - \gamma_5)\not{q}]_{\alpha\beta}$.

Helicity is the projection of the spin, : $\hat{J}_{12}$: (where $J_{ij} = \int d^3x M_{0ij}$), along the three-momentum: $\langle p, \lambda|$: J_{12} : $|0, \lambda\rangle$. The orbital term vanishes for $q^\mu = q^+ \delta_{\mu+}$.

To find projections of spin on 3-directions, we look to

$$u^+(q, \lambda)(\frac{1}{2}\sigma_{ij}) u(q, \lambda),$$
$$v^+(q, \lambda)(-\frac{1}{2}\sigma_{ij}) v(q, \lambda),$$

since $J_3 = \frac{1}{2}\sigma_{12}$,

$$u(q,\lambda) = \lambda u(q,\lambda), \tag{2.9.2}$$
$$v(q,\lambda) = -\lambda v(q,\lambda). \tag{2.9.3}$$

From ((2.9.3)), $J_3 = \frac{1}{2}\gamma_1\gamma_2$,

$$\begin{aligned} 0 &= \gamma^+\gamma^- W(q,\lambda), \\ &= \frac{1}{2}(\gamma_0^2 - \gamma_3^2 + [\gamma_3,\gamma_0])W, \\ &= (1+\gamma_3\gamma_0)W(q,\lambda), \\ W(q,\lambda) &= \gamma_0\gamma_3 W(q,\lambda), \\ J_3 W(q,\lambda) &= \frac{1}{2}\gamma_1\gamma_2\gamma_3\gamma_0 W(q,\lambda), \\ \frac{1}{2}\gamma_5 u &= +\gamma u, \\ \frac{1}{2}\gamma_5 v &= +\gamma v. \end{aligned}$$

The projection matrices for spin along the direction defined byq is given by λ, such as on table 2.3.

$\frac{1}{2}(1+\gamma_5)$	$u_{+\frac{1}{2}}$	$= u_{\frac{1}{2}}$	RH
$\frac{1}{2}(1+\gamma_5)$	$u_{-\frac{1}{2}}$	$= 0$	RH
$\frac{1}{2}(1-\gamma_5)$	$u_{+\frac{1}{2}}$	$= 0$	LH
$\frac{1}{2}(1-\gamma_5)$	$u_{-\frac{1}{2}}$	$= u_{-\frac{1}{2}}$	LH
$\frac{1}{2}(1+\gamma_5)$	$v_{+\frac{1}{2}}$	$= 0$	LH
$\frac{1}{2}(1+\gamma_5)$	$v_{-\frac{1}{2}}$	$= v_{-\frac{1}{2}}$	LH
$\frac{1}{2}(1-\gamma_5)$	$v_{+\frac{1}{2}}$	$= v_{\frac{1}{2}}$	RH
$\frac{1}{2}(1-\gamma_5)$	$v_{-\frac{1}{2}}$	$= 0$	RH

Table 2.3: Helicity of spinors

Hence, for any field we have $(1 \pm \gamma_5)\psi = \psi_{L,R}$, and the Wigner's solution are $\frac{1}{2}\gamma_5 u(q,\lambda) = \lambda u(q,\lambda)$.

2.10 Grassmanian Variables

To construct a path integral for fermions, we now define the *Grassmanian variables*, $\{a_1, a_2, a_3, ..., a_n\}$, analog to a large set of anti-commuting fields

$\psi_i(x, x_0)$. The classical limit of the creation and annihilation operators, $a, a^\dagger$, obeys the Grassmann algebra,

- $a_i a_j = -a_j a_i$, implying that they are nilpotent $a^2 = 0$.
- $a_i z = z a_i$, if $z \epsilon \mathbb{C}$.
- If there are n Grassmann variables, the most general element of the algebra is

$$f(a_i) = \sum_{\{\delta_i = 0,1\}} Z_{\delta_1,\delta_2,\ldots,\delta_n} a_1^{\delta_1} a_2^{\delta_2} \ldots a_n^{\delta_n}.$$

  For example, $f(a_1, a_2) = Z_{010} + Z_{110} a_1 + Z_{011} a_2 + Z_{111} a_1 a_2 = f_{even} + f_{odd}$.

- The Grassmanian calculus is given by defining the derivatives:

$$\begin{aligned} \frac{d}{da_j} a_i &= \delta_{ij}, \\ \frac{d}{da_i} z &= 0, \end{aligned}$$

- One has the propriety $\frac{d}{da_i}\Big(a_j f(a)\Big) - \delta_{ij} f(a) = a_j \frac{df(a)}{da_i}$, i.e., $\{a_i, \frac{d}{da_j}\} = 0$, if $i \neq j$.
- $\{\frac{d}{da_i}, \frac{d}{da_j}\} = 0$.
- There is no second derivatives and no unique antiderivative. The integral is equal to the derivative: $\int da_i = \frac{d}{da_i}$, their actions are the same.
- One can shift the intervals: $\int da_i f(a_i) = \int da_i f(a_i - b_i)$.
- $\int da_i a_j = \delta_{ij} - a_j \int da_i$.
- The integration by parts gives $\int da_j \frac{df(a)}{da_j} g(a) = -\int da_j \{[f(a)_e - f(a)_o] \frac{dg(q)}{da_j}\}$.
- The sign changes when the derivative of $f(a)$ acts on an odd element of the algebra.
- It is possible to construct a mixed integral, $I = \prod_{i,j} \int dy_i \int \xi_j f(y_i, \xi_j)$, where y_i is the commuting number and ξ is the anticommuting one.

- It is possible to construct a gaussian integral, a finite dimensional version of the path version of the path integral, introducing two independent sets of anticommuting $\psi_i, \bar{\psi}_i$, such as

$$\prod_{i=1}^{n} \int d\psi_i d\bar{\psi}_i = \int d\psi_n d\bar{\psi}_n \int d\psi_{n-1} d\bar{\psi}_{n-1} ... \int \psi_1 d\bar{\psi}_1.$$

A general gaussian integral in these variables will produce:

$$\begin{aligned} I_n[M] &= \prod_{i=1}^{n} \int d\psi_i d\bar{\psi}_i e^{-\bar{\psi}_k M_{jk} \psi_k} \\ &= \det M \end{aligned}$$

and with a source term

$$\begin{aligned} I_n[M] &= \prod_{i=1}^{n} \int d\psi_i d\bar{\psi}_i e^{-\bar{\psi}_k M_{jk} \psi_k - \bar{k}_j \psi_j - \bar{\psi}_j k_j} \\ &= \det M e^{-\bar{k}_i M_{ij} k_j}. \end{aligned}$$

2.11 Discrete Symmetries

Discrete symmetry are extensions of the Poincare group that are impossible to obtain by continuous transformations: *parity transformations* P and *time reversal transformation* π.

Dirac Equation and Discrete Symmetries

$$\begin{aligned} \text{Dirac Equation} &\rightarrow (i[\gamma^0 \partial_0 + \gamma \nabla] - m)\psi(x^0, x) = 0 \\ \text{Parity} &\rightarrow (i[\gamma^0 \partial_0 - \gamma \nabla] - m)\psi^P(x^0, x) = 0 \\ \text{Time Reversal} &\rightarrow (i[-\gamma^0 \partial_0 + \gamma \nabla] - m)\psi^T(x^0, x) = 0 \\ \text{Time Reversal} &\rightarrow (i[\gamma^0 \partial_0 + \gamma \nabla] + m)\psi^C(x^0, x) = 0 \end{aligned}$$

Dirac and its Variants

The symmetries of the Dirac equations are:

- **P)** γ^0

$$\begin{aligned} &(i\partial\!\!\!/ - m)\psi(\bar{x}, x^0) = 0, \\ &\gamma^0 \gamma^\mu \gamma^0 = \gamma^\mu \\ &(i\partial\!\!\!/^P - m)\gamma^0 \psi(x, x^0) = 0, \end{aligned}$$

therefore one has $\psi^P(-x, x_0) = \gamma^0 \psi(x, x^0)$.

- **T)** $i\gamma^1\gamma^3 = \sigma^{13}$

$$\begin{aligned}
&(i\partial\!\!\!/ - m)^* \psi(x, x^0) = 0, \\
&T(\gamma^\mu)^* T^{-1} = \gamma^\mu, \\
&(i\partial\!\!\!/^{RT} - m) T \psi^*(x, x^0) = 0,
\end{aligned}$$

therefore one has $\psi^T(x, -x) = T\psi^*(x, x^0)$.

- **C)** $i\gamma^3\gamma^0 = \sigma^{20}$

$$\begin{aligned}
&(i\partial\!\!\!/ - m)^* \psi(x, x^0) = 0, \\
&C\gamma^\mu C^{-1} = -(\gamma^\mu)^T \\
&(i\partial\!\!\!/ - m) T \psi^L(x, x^0) = 0,
\end{aligned}$$

therefore one has $\bar{\psi}^C_\beta = -\psi_\alpha$.

The symmetries in the solutions (spin basis.) u's and v's are:

- **P)** γ^0

$$\begin{aligned}
\gamma^0 u(p, s) &= u(-p, s), \\
\gamma^0 v(p, s) &= -v(-p, s).
\end{aligned}$$

- **T)** $i\gamma^1\gamma^3 = \sigma^{13}$

$$\begin{aligned}
Tu^*(p, s) i(-1)^{s+\frac{1}{2}} u(-p, -s), \\
Tv^*(p, s) = i(-1)^{s-\frac{1}{2}} v(-p, -s).
\end{aligned}$$

- **C)** $i\gamma^3\gamma^0 = \sigma^{20}$

$$\begin{aligned}
\bar{v}_\beta(p, s) &= (-1)^{s+\frac{1}{2}} C_{\beta\alpha} u_\alpha(p, s), \\
\bar{u}_\beta(p, s) &= (-1)^{s+\frac{1}{2}} C_{\beta\alpha} v_\alpha(p, s).
\end{aligned}$$

The symmetries in the vectors are:

- **P)**

$$u_p\psi(x,x^0)u_p^\dagger = \gamma^0\psi(-x,x^0),$$
$$u_pA^\mu(x,x^0)u_p^\dagger = A_\mu(-x,x^0).$$

- **T)**

$$v_T\psi(x,x^0)v_T^\dagger = T\psi(x,-x^0),$$
$$v_TA^\mu(x,x^0)v_T^{-1} = A_\mu(x,x^0).$$

- **C)**

$$u_c\bar{\psi}_\alpha(x,x_0)u_c^{-1} = -\psi_\beta(x^0,x)C^T_{\alpha\beta},$$
$$u_cA^\mu(x,x_\beta)u_c^\dagger = -A,$$
$$\text{where } C^T = C^{-1}, U_cU_c^\dagger = 1.$$

The action on b's, d's, a's is:

- **P)**

$$u_pb(k,s)u_p^{-1} = b(-k,s),$$
$$u_pd(k,s)u_p^{-1} = -d(-k,s).$$

- **T)**

$$v_Tb(p,s)v_T^\dagger = -i(-1)^{-s-\frac{1}{2}}b(-p,s),$$
$$v_Td^\dagger(p,s)v_T^\dagger = i(-1)^{-s-\frac{1}{2}}d^T(-p,-s).$$

- **C)**

$$u_cd(p,s)u_c^{-1} = -(-1)^{s+\frac{1}{2}}b(p,s),$$
$$u_cb(p,s)u_c^{-1} = -(-1)^{s+\frac{1}{2}}d(p,s).$$

2.12 Chirality

The spin of a particle may be used to define a *chirality* for this particle and the invariance of the action of *parity* on a Dirac fermion is called *chiral symmetry*. *Vector Gauge* **theories with massless Dirac fermion field**

ψ **has chiral symmetry**, which means that rotating left-handed and the right-handed components independently makes no difference (U(1) transformation),

$$\psi_L \to e^{i\theta_L}\psi_L \text{ and } \psi_R \to \psi_R, \tag{2.12.1}$$
$$\psi_R \to e^{i\theta_R}\psi_L \text{ and } \psi_L \to \psi_L. \tag{2.12.2}$$

Let us show it explicitly. For a massless fermion, equation 2.1.2 becomes

$$H\psi = \alpha^i P_i \psi, \tag{2.12.3}$$

and we lose one constraint in β. This means we can have a complete basis only using the Pauli matrices. Now, let us define, for instance, the spinor solutions for the free particle, the four-vector given by

$$\psi = u(P)e^{-P^i x_i} = \begin{pmatrix} \chi^s \\ \phi^r \end{pmatrix} e^{-P^i x_i},$$

with $s, r = 1, 2$ and with, for instance,

$$\chi^1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \chi^2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Making $\alpha^i = \pm\sigma^i$ in 2.12.3, we decouple the massless Dirac equation into two equations for two component spinors

$$E\chi = -\sigma^i P_i \chi, \tag{2.12.4}$$
$$E\phi = +\sigma^i P_i \phi. \tag{2.12.5}$$

For example, for solutions on shell on the first of these equations and for positive energy solution, we have $E = |P|$, satisfying

$$\sigma^i P_i \chi = -\chi.$$

In this case, χ is the *left-handed* particle (negative *helicity*). [2] The negative energy solution, $\sigma^i(-P_i)\chi = \chi$, will be the *right-handed* particle (positive *helicity*). We can see clearly that applying a suitable form of 2.12.2 in 2.12.5 will conserve the chirality. [3]

[2] In the extreme relativistic limit, the chirality operator is equal to the helicity operator.

[3] Chirality for massless fermion particles has an important application in the cases of neutrinos. Experimental results show that all neutrinos have left-handed helicities and all antineutrinos have right-handed helicities. In the massless limit, it means that only one of two possible chiralities is observed for either particle. The existence of nonzero neutrino masses complicates the situation since chirality of a massive particle is not a constant of motion. We them work with *Majorana* neutrinos, making they be their own anti-particles.

Let us go even further and see the chirality in massless fermions in the language of the continuity equation $\partial_\mu j^\mu = 0$. The current can be written as $j^\mu = \bar{\psi}\gamma^\mu\psi$ and it is always conserved ($d_\mu j^\mu = 0$) when ψ satisfies the Dirac equation. When coupling the Dirac field to the electromagnetic field, j^μ is just the electric current density.

However, when using our resources of the Dirac algebra we can define one more current,

$$j^{\mu 5} = \bar{\psi}\gamma^\mu\gamma^5\psi. \tag{2.12.6}$$

Now, $\partial_\mu j^{\mu 5} = 2im\bar{\psi}\gamma^5\psi$ and only if $m = 0$ this (axial vector) current is also conserved. It is then the electric current of left-handed and right-handed particles, and separately conserved.

It is beautiful to see that both currents are just the *Noether currents* for our symmetries defined in 2.12.2:

$$\psi(x) \to e^{i\alpha}\psi(x) \text{ and } \psi(x) \to e^{i\alpha\gamma^5}\psi(x),$$

where the first is a symmetry of the Dirac lagrangian and the second, the chiral transformation, is a symmetry of only the derivative term of the lagrangian, not the mass term, i.e. it conserves only for $m = 0$.

In conclusion, massive fermions do not exhibit chiral symmetry since the mass term in the Lagrangian, $m\bar{\psi}\psi$, breaks chiral symmetry [4].

From the same argument we can clearly see that the *quantum electrodynamics theory* is invariant under parity (together with the derivative-dependent term of electromagnetic interaction, $\bar{\psi}\gamma_\mu\psi A^\mu$). We can heuristically see that its action is invariant and the quantization is also invariant. The invariance of the action follows from the classical invariance of Maxwell's equations. The invariance of the canonical quantization procedure can be seen by the fact that vector bosons can be shown to have odd intrinsic parity, and all axial-vectors to have even intrinsic parity.

2.13 The Parity Operators

If $\psi(x)$ solves the Dirac equation, so does $\gamma^0\psi(x) = \psi(x^0, -x)$ [5]. Therefore, a fifth gamma matrix is defined,

$$\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3 = \frac{i}{4!}\epsilon^{\mu\nu\lambda\sigma}\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma,$$

[4] Spontaneous chiral symmetry breaking also occur in some theories to make the field acquires the mass.

[5] Now we are again using the metric signature given by $(+ - - -)$.

which commutates to all other γ-matrices,

$$\{\gamma^5, \gamma^\mu\} = 0.$$

From this new matrix γ^5, one can define the projective operators, $P_\pm$, that project out states in the four-solutions ψ in our previous Dirac-Pauli representation and also in the Weyl representation,

$$\gamma^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}, \gamma^0 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}, \gamma^5 = \begin{pmatrix} -I & 0 \\ 0 & I \end{pmatrix}.$$

It is easier to see the projection in the Weyl (or *chirality representation*), where the projector operator just flips the spinor in the appropriate direction. For example, for the free-particle solution of the previous exercise, putting it back in 2.1.3, we can eliminate the exponential dependence and work only with the spinor part on χ^s and ϕ^r. We can then see explicitly the action of the projector operator:

$$P_- = \frac{1}{2}(1-\gamma_5)\psi = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \chi^s \\ \phi^r \end{pmatrix} = \begin{pmatrix} \chi^s \\ 0 \end{pmatrix} \rightarrow Left - handed.$$

$$P_+ = \frac{1}{2}(1+\gamma_5)\psi = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \chi^s \\ \phi^r \end{pmatrix} = \begin{pmatrix} 0 \\ \phi^r \end{pmatrix} \rightarrow Right - handed.$$

For the sake of completeness in analyzing the parity in the Dirac equation, the parity operator for the Dirac equation is given by γ^0,

$$\begin{aligned} &(i\partial\!\!\!/ - m)\psi(x^0, x) = 0, \\ &\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \rightarrow \gamma^\mu(i\partial\!\!\!/^P - m)\gamma^0\psi(x^0, x) = 0, \end{aligned}$$

where we use the notation $A\!\!\!/ = \gamma_\mu A^\mu$. Therefore, one has $\psi^P(-x, x_0) = \gamma^0\psi(x, x^0)$.

2.14 Propagators in The Field

Scalar Field Propagator

The scalar theory for the free field is given by the hamiltonian (density)

$$\mathcal{H} = \frac{1}{2}\Pi^2 + \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}m^2\phi^2, \tag{2.14.1}$$

and the respectively lagrangian (density),

$$\mathcal{L} = -\frac{1}{2}(\partial^\mu\phi)^2 - \frac{1}{2}m^2\phi. \tag{2.14.2}$$

The *Feynman propagator* is then

$$\Delta(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{e^{ik(x-y)}}{k^2+m^2-i\epsilon}. \tag{2.14.3}$$

This propagator is the *Green's function* for the Klein-Gordon equation, i.e. a solution of this equation (taking $\epsilon \to 0$),

$$(-\partial_x^2 + m^2)\Delta(x-y) = \delta^4(x-y). \tag{2.14.4}$$

However we can evaluate $\Delta(x-y)$ explicitly by taking the k^0 integral in 2.14.3, which is a contour integral in the complex k^0 plane, where the 4-vector inner product is $k(x-y) = k^0(x^0-y^0) - \vec{k}(\vec{x}-\vec{y})$ (in the Minkowski spacetime, this expression is not uniquely defined because of the poles, $k^0 = \pm\sqrt{\vec{p}^2+m^2}$).

In resume, the different choices of how to deform the integration contour lead to different sign for the propagator. Let us them calculate it explicitly by the residue theorem. We choose the causal (retarded) propagator, as in the contour in figure 2.14. The integral is zero if x or y are spacelike (if $x^0 > y^0$, i.e. x^0 is future of y^0). The integral is then

$$\begin{aligned} I &= \lim_{\epsilon\to 0} \int_{-\infty}^{\infty} \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{dk^0}{2\pi} \frac{e^{ik(x-y)}}{-\vec{k}^2 - m^2 - (k^0+i\epsilon)^2}, \\ &= i\theta(t-t') \int dk^3 e^{ik(x-y)} + i\theta(t-t') \int d^3k e^{-ik(x-y)}. \end{aligned}$$

Using the formalism of *functional integrals* (writing the path integral), we can evaluate the ground-state expectation value of the time-ordered product of our scalar fields in terms of this propagator,

$$\langle 0|T\phi(x_1)\phi(x_2)|0\rangle = -i\Delta(x_2-x_1). \tag{2.14.5}$$

A little comment for future discussions in the Feynman diagrams is that the result in 2.14.5 is generalized for many fields by the *Wick's Theorem*,

$$\langle 0|T\phi(x_1)...\phi(x_{2n})|0\rangle = -i^n \sum_{pairs} \Delta(x_{i_1}-x_{i_2})...\Delta(x_{i_{2n-1}}-x_{i_{2n}}). \tag{2.14.6}$$

Now that we understand the propagator for the scalar field, we can compute it for the Dirac field.

Free Fermion Propagator

We are going to consider the free Dirac field,

$$\psi(x) = \sum_{s=1,2} \int d^3p \Big[b_s(p) u_s(p) e^{ipx} + d_s^\dagger(p) v_s e^{-ipx} \Big], \tag{2.14.7}$$

$$\bar{\psi}(y) = \sum_{s=1,2} \int d^3p' \Big[b_{s'}(p') \bar{u}_{s'}(p') e^{ip'y} + d_{s'}^\dagger(p') v_{s'} e^{-ip'y} \Big], \tag{2.14.8}$$

where we sum on the spin polarization. The annihilation operators are given by

$$b_s(p)|0\rangle = d_s(p)|0\rangle = 0, \tag{2.14.9}$$

and the anticommutation relations are

$$\{b_s(p), b_{s'}^\dagger(p')\} = (2\pi)^3 \delta^2(p-p') 2\omega \delta_{ss'}, \tag{2.14.10}$$

$$\{d_s(p), d_{s'}^\dagger(p')\} = (2\pi)^3 \delta^2(p-p') 2\omega \delta_{ss'}, \tag{2.14.11}$$

with zero to all other combinations.

Now we want to compute the Feynman propagator

$$S(x-y)_{\alpha\beta} = \theta(x^0 - y^0) \langle 0 | \{ \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle, \tag{2.14.12}$$

$$= i \langle 0 | T \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle, \tag{2.14.13}$$

where $\theta(t)$ is the step function and T is the time-ordered product,

$$T\psi_\alpha(x)\bar{\psi}_\beta(y) = \theta(x^0 - y^0)\psi_\alpha(x)\bar{\psi}_\beta(y) - \theta(y^0 - x^0)\bar{\psi}_\beta(y)\psi_\alpha(x) \tag{2.14.14}$$

The minus sign in the last result comes from the anticommutation propriety, $\{\psi_\alpha(x), \bar{\psi}_\beta(y)\} = 0$ when $x^0 \neq y^0$, and we will discuss the causality in the end. To derive the propagator and show that it obeys causality, let us now insert 2.14.8 into 2.14.14,

$$\begin{aligned} \langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle &= \sum_{s,s'} \int d^3p d^3p' e^{ipx - ip'y} u_s(p)_\alpha \bar{u}_{s'}(p')_\beta \langle 0 | b_s(p) b_{s'}^\dagger(p') | 0 \rangle, \\ &= \sum_{s,s'} \int d^3p d^3p' e^{ipx - ip'y} u_s(p)_\alpha \bar{u}_{s'}(p') (2\pi)^3 \delta^3(p-p') 2\omega \delta_{ss'} \\ &= \sum_{s,s'} \int d^3p e^{ip(x-y)} u_s(p)_\alpha \bar{u}_{s'}(p')_\beta . \end{aligned}$$

Using the result of the sum of all spin polarizations,

$$\sum_s u_s(p)\bar{u}_s(p) = -\not{p} + m, \tag{2.14.15}$$

$$\sum_s v_s(p)\bar{v}_s(p) = -\not{p} - m, \tag{2.14.16}$$

we finally have

$$\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = \int d^3p e^{ip(x-y)}(-\not{p}+m)_{\alpha\beta}.$$

In the same fashion,

$$\begin{aligned}
\langle 0|\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle &= \sum_{s,s'}\int d^3p d^3p' e^{-ipx+ip'y} v_s(p)_\alpha \bar{v}_{s'}(p')_\beta \langle 0|d_s(p)d^\dagger_{s'}(p')|0\rangle,\\
&= \sum_{s,s'}\int d^3p d^3p' e^{-ipx+ip'y} v_s(p)_\alpha \bar{v}_{s'}(p')(2\pi)^3\delta^3(p-p')2\omega\delta_{ss'},\\
&= \sum_{s,s'}\int d^3p e^{-ip(x-y)} v_s(p)_\alpha \bar{v}_{s'}(p')_\beta,\\
&= \int d^3p e^{-ip(x-y)}(-\not{p}-m)_{\alpha\beta}.
\end{aligned}$$

These two results can be combined in the time-ordered product, and we get

$$\langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)|0\rangle = -i\int \frac{d^4p}{(2\pi)^4} e^{i(p(x-y)} \frac{(-\not{p}+m)_{\alpha\beta}}{p^2+m^2-i\epsilon} = iS(x-y)_{\alpha\beta},$$

recovering the Feynman propagator. This is of course the inverse of the Dirac operator

$$(-i\not{\partial}+m)S(x-y) = \delta^4(x-y).$$

Again, let us consider the vacuum expectation value of a time-ordered product of more than two fields. We must have an equal number of ψ and $\bar{\psi}$ to get a nonzero result. Concerning the statistics, there is an extra minus sign if the ordering of the fields in their pairs is odd permutation of the original ordering. For instance,

$$\begin{aligned}
\langle 0|T\psi_\alpha(x)\bar{\psi}_\beta(y)\psi_\gamma(z)\bar{\psi}_\delta(w)|0\rangle = -i^2\Big[&S(x-y)_{\alpha\beta}S(z-w)_{\gamma\delta}\\
&-S(x-w)_{\alpha\delta}S(z-y)_{\gamma\beta}\Big],
\end{aligned}$$

Let us analyze these results. We recognize the right side integral of our derivations to be the commutator of two scalar fields $\Delta_{KG}(x-y)$, hence the anticommutator of the Dirac theory is

$$i\Delta_{\alpha\beta}(x-y) = (-i\partial\!\!\!/ + m)i\Delta_{KG}(x-y),$$

and it will vanishes since $\Delta_{KG}(x-y)$ vanishes at spacelike separations, concluding that Dirac theory is **causal**.

From reference [PS1995], we see that if we had quantized the Dirac theory with commutators instead of anticommutators, we would have a violation of causality, with the exponentials of 2.14.8 summing up instead of having a minus sign and we would have a decaying at equal time and long distances ($\tilde{\Delta} \to \frac{e^{mx}}{x^2}, mx \to \infty$). In another words, if the theory were to be quantized with commutators, the field operators would not commute at equal time at distances shorter than the Compton wavelength, violating the causality.

We had just obtained the *Spin-Statistic Theorem*, , which states that fields with half-integer (integer) spin must be quantized as fermions (bosons), making use of anticommutators (commutators). If the theory is quantized with the wrong spin-statistic connection, it either becomes non-local (no causality) or it has no ground state (spectrum with negative norm).

Chapter 3

Non-Abelian Field Theories

Let us generalize the concept of local phase invariance to the Lagrangian densities with N identical spinor fields that posse global symmetry $U(N)$,

$$\psi_i'(x) = U_{ij}\psi_j(x).$$

If $U \to U(x)$, the Dirac Lagrangian is no longer invariant. For $N = 1$, it is possible to recover the QED case, (2.8.4), however,, for $N > 1$, the changes on the Lagrangian are only canceled if one introduces *nonabelian gauge fields*, also known as *Yang-Mills fields*, $(A^\mu)_{ij}$.

Starting with Lagrangians with global spin $U(N) = U(1) \times SU(N)$, one has a set of N Dirac fields ψ_i, with mass m_ψ and N scalars fields ϕ_i with mass m_ϕ. The gauge fields are $N \times N$ matrices,

$$(A^\mu)_{ij} = \sum_{a=1}^{N^2-1} (T_a)_{ij} A_a^\mu(x)\bar{\psi}, \tag{3.0.1}$$

where T_a are the generators of $SU(N)$ in the $N \times N$ (fundamental) representation. For this expression to be consistent, A^μ should be an element of the Lie algebra, where the number of gauge fields is the dimension of the group. The normalization is chosen to be

$$\text{tr } (T_a T_b) = \frac{1}{2}\delta_{ab}.$$

The covariant derivative is

$$(D_\mu[A])_{ij} = \delta_{ij}\partial_\mu + ig(A_\mu)_{ij}, \tag{3.0.2}$$

$$D = (\not{\partial} + i\epsilon\not{A}). \tag{3.0.3}$$

The field strength (also a matrix) is

$$\begin{aligned} F_{\mu\nu} &= \delta_\mu A_\nu - \partial_\nu A_\mu + ig[A_\mu, A_\nu], \\ &= \sum_a^{N^2-1} F_{\mu\nu,a} T_a. \end{aligned}$$

The Yang-Mills Lagrangian is

$$\mathcal{L}_{YM} = \bar{\psi}_i \{[iD^\mu(A)]_{ij}\gamma_\mu - m\delta_{ij}\}\psi_j, \tag{3.0.4}$$

where the covariant derivative was defined in (3.0.3). One can make global invariance (phase $U(1)$ and global $SU(N)$) by local combining it to the vector fields gauge theories. A resume of the simplest Yang-Mill's theory factors can be seen in the table 3.1.

COMPONENT	YANG-MILLS
Fields	Dirac ψ_i, Scalar ϕ_i
Generators	$T_a, a = 1, ..., n^2 - 1$
Commutator	$[T_a, T_b] = if_{abc}T_c$
Matrix to the Fields	$A^\mu(x)_{ij} = \sum_{a=1}^{n^2-1} Aa_a^\mu(x)(T_a^R)_{ij}$
Covariant Derivative	$D^\mu[A]_{ij} = \delta_{ij}\partial^\mu + igA^\mu(x)_{ij}$
Field Strength	$F_{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu + ig[A^\mu, A^\nu]$
Fields in components	$F_{\mu\nu} = \sum_{a=1}^{n^2-1} F_{\mu\nu,a}T_a^R$

Table 3.1: The Yang-Mills theory.

3.1 Gauge Transformations

The infinitesimal gauge transformations of the vectors A^μ are given by

$$\begin{aligned} U^R(x) &= e^{ig\sum_{a-1}^{N^2-1}\Lambda_a(x)T_a^R}, \\ &= 1 + ig\sum_{a=1}^{N^2-1}\delta\Lambda_a(x)T_a, \\ &= [1 + ig\delta\Lambda(x)]_{ij}. \end{aligned}$$

The A^μ transformations are constructed in a way to give the form invariant Lagrangian,

$$\text{Finite } \rightarrow A'^\mu = UA^\mu U^{-1} + \frac{i}{g}(\partial^\mu U)U^{-1}$$

$$\text{Infinitesimal } \rightarrow A'^\mu = A^\mu - \partial^\mu \delta\Lambda(x) + ig[\delta\lambda(x), A_\mu(x)],$$

where the first two terms are the same as in any theory with $U(1)$ gauge invariance and the last term is new for non-abelian theories. To complete the Lagrangian (3.0.4), we need to supply a new term which is a generalization of the Maxwell field density, this field strength transforms covariantly as

$$F'_{\mu\nu}(x) = U(x)F_{\mu\nu}U^{-1},$$

with the following components, where the last term is Yang-Mills field,

$$\begin{aligned} F_{\mu\nu} &= U(x)F_{\mu\nu,a}t_a, \\ F'_{\mu\nu,a} &= \partial_\mu A_{\mu,a} - \partial_\nu A_{\mu,a} - gC_{abc}A_{\mu,c}A_{\nu,c}, \end{aligned}$$

The gauge invariant Lagrangian is then

$$\begin{aligned} \mathcal{L}_{YM} &= -\frac{1}{4}\sum_a F_{\mu\nu,a}F_a^{\mu\nu}, \\ &= \frac{1}{2}\,\mathrm{tr}[F_{\mu\nu}F^{\mu\nu}]. \end{aligned}$$

Putting all this together, the Lagrangian invariance under $SU(N)$ for the spinor is and a scalar is

$$\mathcal{L}_{SU(N)} = |D_\mu(A)_{ij}\phi_k|^2 - m_\phi^2|\phi|^2 + \bar{\psi}_i(i\not{D}[A]_{ij} - m_\phi\delta_{ij})\psi_j - \frac{1}{2}\,\mathrm{Tr}\,[F_{\mu\nu}, F^{\mu\nu}].$$

3.2 Lie Algebras

In *compact Lie Algebras*, that are the interest here, the number of generators T^a is finite. Any infinitesimal group element$g$ can be written as

$$g(\alpha) = 1 + i\alpha^a T^a + \mathcal{O}(\alpha^2),$$

where

$$[T^a, T^b] = if^{abc}T^c \tag{3.2.1}$$

are the non-abelian generators commutation relation. If one of the generators commutes with all of the others, it generates an independent continuous abelian group $\psi \to e^{i\alpha}\psi$, called $U(1)$. If the algebra contains such commuting elements, it is a semi-simple algebra. Finally, if the algebra cannot be divided into two mutually commuting sets, it is simple. The condition that a Lie Algebra is compact and simple restricts it to four infinity families and 5 exceptions.

Unitary Transformations of N-dimensional vectors For η, ξ n-vectors, with linear transformations $\eta_a \to U_{ab}\eta_b$ and $\xi_a \to U_{ab}\xi_b$, this subgroup preserves the unitary of these transformations, i.e. it preserves $\eta_a^*\xi_a$. The pure phase transformation $\xi_a e^{i\alpha}\xi_a$ is removed to form $SU(N)$, consisting of all $N \times N$ unitary transformations satisfying $\det(U) = 1$. The $N^2 - 1$ generators of the group are the $N \times N$ matrices T^a under the condition $\mathrm{tr}[T^a] = 0$.

Orthogonal Transformations of N-dimensional vectors The subgroup of unitary $N \times N$ transformations that preserves the symmetric inner product: $\eta_a E_{ab} \xi_b$ with $E_{ab} = \delta_{ab}$, which is the usual vector product, so this is the rotation group in N dimensions, $SO(N)$, or the rotation group in $2n+1$ dimensions, $SU(2n+1)$. There is an independent rotation to each plane in N dimensions, thus the number of generators are $\frac{N(N-1)}{2}$.

Symplectic Transformations of N-dimensional vectors The subgroup of unitary $N \times N$ transformations. For N even, it preserves the anti-symmetric inner product $\eta_a E_{ab} \xi_b$,

$$E_{ab} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

where the elements of the matrix are $\frac{N}{2} \times \frac{N}{2}$ blocks, $Sl(N)$, with $\frac{N(N+1}{2}$.

Representations

If the Lie algebra is semi-simple, the matrices t_r^a are traceless and the trace of two generator matrices are positive definite given by

$$\mathrm{tr}\ [T_r^a, T_r^b] = D^{ab}.$$

Choosing a basis for T^a which has $D^{ab} \propto \mathcal{I}$ for one representation means that it will be true for all representations:

$$\text{tr } [T_r^a, T_r^b] = C(r)\delta ab. \tag{3.2.2}$$

From the commutation relations, one can write the totally anti-symmetric structure constant as

$$f^{abc} = -\frac{i}{C(r)} \text{ tr } [t_r^a, t_r^b] t_r^c. \tag{3.2.3}$$

For each irrep r of G, there will be a conjugate $\bar{r}$ given by

$$t_r^a = -(t_r^a)^* = -(t_r^a)^T, \tag{3.2.4}$$

if $\bar{r} \sim r$ then $t_{\bar{r}}^a = U t_r^a U^\dagger$ and the representation is real. The two most important irreducible representations are the *fundamental* and the *adjoint*, which dimensions are given by table 3.2.

Fundamental In $SU(N)$, the basic irrep is the N-dimensional complex vector, and for $N > 2$, this irrep is complex. In $SO(N)$ it is real and in $Sl(N)$ it is pseudo-real.

Adjoint This is the representation of the generators, $r = G$, and the representation's matrices are given by the structure constants $(t_c^b)_{ab} = if^{abc}$ where $([t_G^b, t_a^c])_{ae} = if^{bcd}(t_G^d)_{ae}$. Since the structure constants are real and anti-symmetric, this irrep is always real.

	FUNDAMENTAL	ADJOINT	EXAMPLES
$SU(N)$	N, complex	$N^2 - 1$	N=4, 4 and 15
$SO(N)$	N, real	$\frac{N(N-1)}{2}$	N=4, 4 and 6
$Sl(N)$	N, pseudo-real	$\frac{N(N+1)}{2}$	N=4, 4 and 10

Table 3.2: Dimensions of the most important irreps of the compact and simple Lie algebras.

A good way of seeing the direct application of this theory on fields is, for example, the covariant derivative acting on a field in the adjoint representation,

$$\begin{aligned} (D_\mu \phi)_a &= \partial_\mu \phi_a - igA_\mu^a (t_G^b)_{ac} \phi_c, \\ &= \partial_\mu \phi_a + gf^{abc} A_\mu^b \phi_c, \end{aligned}$$

and the vector field transformation

$$A^a_\mu \to A^a_\mu + \frac{1}{g}(D_{\mu\alpha})^a.$$

Casimir Operator

For any simple Lie algebra, the operator $T^2 = T^aT^a$ commutes with all groups of generators:

$$\begin{aligned}
[T^b, T^aT^a] &= (if^{bac}T^a + T^a(if^{bac}T^c), \\
&= if^{bac}\{T^c, T^a\}, \\
&= 0.
\end{aligned}$$

For the adjoint representation,

$$f^{acd}f^{bcd} = C_2(G)g^{ab},$$
$$d(r)C_2(r) = d(G)C(r),$$

where, for example, for $SU(2)$, $C(r) = \frac{1}{2}$. For$SU(N)$, one has the *quadratic Casimir operator coefficient*,

$$C_2(r) = \frac{1}{2}\frac{(N^2-1)}{N}$$

3.3 Spontaneous Symmetry Breaking

So far the locally invariant Lagrangian involved only massless vector matrices. The *spontaneous symmetry breaking* makes it possible to the Lagrangian to acquire at once both local gauge invariance and massive vectors. The system chooses a ground state, with a definitive but arbitrary direction, any choice breaks the symmetry. While the ground states breaks the symmetry, the Lagrangian remains gauge invariant.

Example: The $SU(2) \times U(1)$ Model

In the *Higgs mechanism*, we postulate a scalar field with a internal symmetry $U(1)$ or great, coupled to a gauge field. We choose a doublet of complex scalar fields $\phi_i, i = 1, 2$ and write the Lagrangian as

$$\begin{aligned}
\mathcal{L}_{U(1)} &= (iD_\mu(A)\phi^*)(iD^\mu(A)\phi) - V(|\phi|) - \frac{1}{4}F^2, \\
D_\mu &= \partial_\mu + igA_\mu, \\
F_{\mu\nu} &= \partial_\mu A_\nu - \partial_\nu A_\mu,
\end{aligned}$$

Next step is to write explicitly the

$$V(|\phi|) = -\mu^2|\phi|^2 + \lambda(|\phi|^2)^2,$$

which has its minimum in $|\phi_0|^2 = \frac{\nu^2}{2} = \frac{\mu^2}{2\lambda}$, as in figure (3.1) and (3.2).

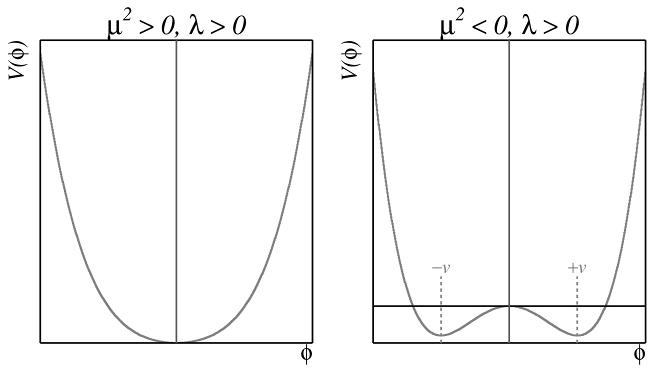

Figure 3.1: The Higgs Potential for a real field.

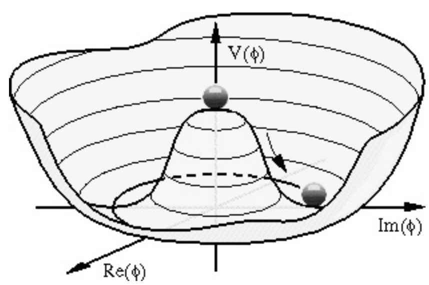

Figure 3.2: The Higgs Potential for a Complex field, in this case the minimum is the circle on $|\phi_0|$.

The vacuum expectation value, $\langle 0|\phi|0\rangle$, is now different of zero and it is not unique. Reparameterizing the scalar field to

$$\phi(x) = e^{i\frac{\xi(x)}{2V}}\frac{V + \eta(x)}{\sqrt{2}}, \tag{3.3.1}$$

where $V = \frac{\mu}{\sqrt{\lambda}}$, one performs the following replacement of the real and imaginary parts of ϕ_i: Reϕ, Imϕ $\rightarrow \xi(x), \eta(x)$ (phase and modulus). The potential is then independent of $\xi(x)$ and one has only to replace $|\phi|$ by (3.3.1) in V. The mass will be generated to $\eta(x)$ while $\xi(x)$ will be massless, being only a physical degree of freedom.

$$\begin{aligned} V(|\phi|^2) &= -\mu^2\left(\frac{V+\eta(x)}{\sqrt{2}}\right)^2 + \lambda\left(\frac{V+\eta(x)}{\sqrt{2}}\right)^4, \\ &= -\frac{\mu^2}{4\lambda} + \mu^2\eta^2(x) + \sqrt{\lambda}\mu\eta^3(x) + \frac{\lambda}{4}\eta^4(x), \end{aligned}$$

where the first term is constant and the third is the interaction. The fields $\xi(x)$ parametrize the position of ϕ around the 3-dimensional minimum while $\eta(x)$ measures the distance from this minimum. The transference of the phase degree of freedom from ϕ to A^μ makes it appears as a mass, as it can be seen at table (3.3)

Start with	Degree of Freedom	End	Degree of Freedom
ϕ	2	$\eta(x)$	1
$A^\mu_{m=0}$	1	$A^\mu_{m\neq 0}$	3
$V(\|\phi\|)$	Gauge Invariant	$V(\eta)$	Gauge Invariant

Table 3.3: Degree of freedom for spontaneous symmetry breaking of fields.

The Lagrangian now has a mass term for the field, which is Proca-type mass. Spontaneous symmetry breaking of any kind implies the existence of a massless scalar, the Goldstone boson. The difference to the Higgs is the this would be a boson absorbed as a gauge field, i.e. vector.

The theory in terms of new fields is simplified by canceling the nonabelian phase by a $SU(2)$ gauge transformation, which is the *Standard Model* of elementary particles is the *electroweak* group representation, with the gauge bosons W^-, W^+, Z^0. This group transformation takes place by

$$\begin{aligned} &\phi' = U\phi, \\ &B'_\mu = UB_\mu U^{-1} + \frac{i}{g}(\partial U)U^{-1}, \\ &U(x) = e^{-i\xi(x)\frac{\sigma}{2v}}. \end{aligned}$$

Now the Lagrangian is independent of the fields ξ and its scalar part is

$$\mathcal{L}_{Higgs} = |iD(B,C)_{ij}\phi_j|^2 - V(|\phi|^2, \tag{3.3.2}$$

$$\text{where } |\phi|^2 = |\phi^\dagger|^2 + |\phi^0|^2, \tag{3.3.3}$$

The potential and its extremal is

$$\begin{aligned} V(|\phi|) &= -\mu^2|\phi|^2 + \lambda|\phi|^4, \\ \left.\frac{dV}{d|\phi|}\right|_{\phi=0} &= -2\mu^2 V + 4\lambda V^3 = 0, \\ V_{min} &= \frac{\mu^2}{2\lambda}. \end{aligned}$$

We write the four degree of freedom of ϕ as

$$\phi(x) = e^{-\xi(x)\frac{\sigma}{2}} \begin{pmatrix} 0 \\ \frac{V+\eta(x)}{\sqrt{2}} \end{pmatrix},$$

where the three ξ are three massive vectors and one η fields remains distances from V_{min}. The Lagrangian, ((3.3.3), is then

$$\mathcal{L}_{Higgs} = \frac{1}{2}[(\partial_\mu \eta)^2 - 2\mu^2\eta^2] - V\lambda\eta^3 + \frac{\lambda}{4}\eta^4 + \frac{V^2}{8}[g^2(B_1^2+B_2^2) + (g'C - gB_3)^2](V+\eta)^2,$$

where the V^2 terms give the Proca mass: $\frac{V^2 g^2 B_{1,2}^2}{8}$, $\frac{V^2(g'C-gB_3)^2}{8}$. The physical particles are then represented by

$$\begin{aligned} W^{\pm} &= \frac{1}{\sqrt{2}}(B_1 \pm iB_2), \\ Z &= -(\sin\theta_W + B_3 \cos\theta_W), \\ \gamma &= (\cos\theta_W + B_3 \sin\theta_W), \\ \text{where } \sin\theta_W &= \frac{g'}{\sqrt{g^2+g'}}. \end{aligned}$$

The masses of the electroweak gauge bosons are then $M_W = \frac{Vg}{2}, M_z = \frac{M_W}{cos\theta_w}$.

Chapter 4

Quantum Electrodynamics

In this section we are going to understand and to derive the Lagrangian for the theory of interaction of photons and matter the quantum electrodynamic, represented by (2.8.4). This theory is given by a vector field

$$A^{\mu}(x) = C^{\mu}\cos\theta_w + B_3^{\mu}\sin\theta_w.$$

After the spontaneous symmetry breaking, the photon Lagrangian is Maxwell-form,

$$\mathcal{L}_A = -\frac{1}{4}F_{\mu\nu}^2(A) - \frac{\lambda}{2}(\partial A)^2.$$

This couples to fermions as

$$\mathcal{L}_{ferm}(\psi_F, A) = \sum \bar{\psi}_F(i(\partial\!\!\!/ - ieQ_F A\!\!\!/) - m_F)\psi_F,$$

where Q_F is the charge of the fermion in units of positron charge ($Q_F = 1, \frac{2}{3}, -\frac{1}{3}, 0$). The $\mathcal{L}_{QED}$ is then $\mathcal{L}_A + \mathcal{L}_{ferm}$.

Because the minimum coupling, the term $\bar{\psi}A\psi$ is not free. As a scalar theory, it makes it necessary to find the Green's function $\langle 0|T(\psi(x), \psi(y)A^{\mu}(z), ...)|0\rangle$ to derive the S-matrix,

$$\mathcal{L}_{QED} \to Z[J^{\mu}(x), k_{\alpha}(y), \bar{k}_{\beta}(z)] \to \text{ S-matrix.}$$

By analogy to the scalars fields, one can write the generating functional for the Green's functions,

$$Z_{QED}[J^{\mu}, k_{\alpha}, \bar{k}_{\beta}] = \frac{W[J, k, \bar{k}]}{W[0,0,0]}, \tag{4.0.1}$$

where W are the path integrals and J is the source. The path integral should be over classical c-numbers for fields.

$$\begin{aligned} W_{QED}[A,k,\bar{k}] &= \int [dA^\mu][d\psi_\alpha][d\bar{\psi}_\beta] e^{i\int d^4x \mathcal{L}_{QED} - i(J,A) - i(k,A) - i(\bar{\psi},k)}, \\ &= \int [dA^\mu] W[A,k,\bar{k}] e^{i\int d^4x [\mathcal{L}(A) - \frac{\lambda}{2}(\partial A)^2]}, \\ &= \int [d\psi][d\psi] e^{i\int d^4x \bar{\psi}[i\not{D}(A) - m]\psi} e^{-i(\psi,k) - i(k,\psi)}, \end{aligned}$$

where the last two exponentials are the fermion Gaussian integral. We recall the Gaussian scalar form,

$$\begin{aligned} \omega(j_i) &= \int_{-\infty}^{\infty} \prod_{i-1}^{n} dx_i e^{-\frac{1}{2}x_i M_{ij} x_J - J_k x_k}, \\ &= \frac{\prod^{n/2}}{\sqrt{\det M}} e^{-\frac{1}{2}J_k (M^{-1})_{kl} J_l}. \end{aligned}$$

The Green's function is then

$$\begin{aligned} G(y_i,x_i) &= \langle 0|T[\prod_{i=1}^{n} \psi_{ai}(y_i)\bar{\psi}_{bi}(x_i)]|0\rangle, \\ &= \prod_{i=1}^{n} \{[i\frac{\delta}{\delta(\bar{k}_{ai},y_i)}][-ik_{bi}(x_i)]\} Z[k,\bar{k},A]|_{k=\bar{k}=0}, \end{aligned}$$

which gives propagator for fermions

$$\begin{aligned} S_F(z) &= \frac{1}{(2\pi)^4}\int d^4k e^{-ikz}\frac{1}{\not{k} - m + i\epsilon}, \\ &= \frac{1}{(2\pi)^4}\int d^4k e^{-ikz}\frac{\not{k} - m}{k^2 - m^2 + i\epsilon}, \end{aligned}$$

with $\not{k}\not{k} = k^2$. The charged scalar is

$$\begin{aligned} W_\psi[L,\bar{L}] &= \int [d\phi][d\phi^*] e^{-iS_{kg} - i(\bar{L},\phi) - i(\phi^*,L)}, \\ &= W[0,0] e^{-i\int_{WZ} L^*(\omega)\Delta(\omega - z)L(z)}, \end{aligned}$$

giving, finally, the generating functional for fermions

$$W_\phi[A,k,\bar{k}] = \int \mathcal{D}\psi\mathcal{D}\bar{\psi} e^{i\int d^4y \bar{\psi}(i\not{D}(A) - m)\psi - i\bar{k}\psi - i\bar{\psi}k},$$

$$Z = \frac{1}{W[0,0,0]}\int \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A e^{i\int d^4y \psi(i\not{D} - m)\psi - \bar{k}\psi - \bar{\psi}k - yA}.$$

4.1 Functional Quantization of Spinors Fields

The Grassmann variables defined at section (2.10) are now used in the fields ψ, defined as a function of spacetime whose values are anticommuting number. We define this field in terms of orthonormal basis functions,

$$\psi(x) = \sum_i \psi_i \phi_i(x),$$

where ψ is the Grassmannian variable and ϕ is the four-component spinor. The Dirac two point correlation function is

$$\langle |T\psi(x_i)\bar{\psi}(x_2)|0\rangle = \frac{\int D\psi D\bar{\psi} e^{i\int d^4x\bar{\psi}(i\gamma^\mu - m)\psi]}\psi(x_1)\bar{\psi}(x_2)}{\int D\bar{\psi} D\psi e^{i\int d^4x\bar{\psi}(i\gamma^\mu - m)\psi}} \tag{4.1.1}$$

Using the Gaussian integrals

$$\begin{aligned}(\prod_i \int d\theta_i^* d\theta_i) e^{-\theta_i^* B_{ij}\theta_j} &= \prod_i b_i \\ &= \det B,\end{aligned}$$

if θ were an ordinary number, one would have $\frac{(q\pi)^n}{detB}$. Also

$$(\prod_i \int d\theta_i^* d\theta_i)\theta_k \theta_l^4 e^{-\theta_i^* B_{ij}\theta_j} = \det B(B^{-1})_{kl}.$$

One can then calculate (4.1.1), where the denominator is $\det(i\gamma^\mu\partial - m)$ and numerator is $\det(i\gamma^\mu\partial - m)\frac{-1}{i\gamma^\mu\partial - m}$. From this is possible to recover the propagator for fermions,

$$\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle = S_F(x_1 - x_2) = \int \frac{d^4k}{(2\pi)^4}\frac{ie^{-ik(x_i - x_2)}}{\not{k} - m + i\epsilon}$$

4.2 Path Integral for QED

The QED Lagrangian with sources is

$$\mathcal{L}_{QED} = \sum_{flavors} \bar{\psi}(i\not{D}[A] - m)\psi - \frac{1}{4}F^2(A) - \frac{\lambda}{2}(\partial A)^2.$$

Writing it considering only connected diagrams, i.e all lines connected to some external point in the Green's function, we get

$$\begin{aligned} Z_{free}[J_\mu, k_\alpha, \bar{k}_\alpha] &= \frac{W[J,k,\bar{k}]}{W[0,0,0]}, \\ &= e^{-\int_{w,z}[\frac{1}{2}J^\mu(w)G_{\mu\nu}(w-z)J^\nu(z) + \bar{k}_\alpha(w)S_{\alpha\beta}(w-z)k_\beta(z)]}, \end{aligned}$$

where

$$W[J,k,\bar{k}] = W_{free}e^{-\int d^4y[-\delta_k(y)]_\alpha(\gamma^\mu)_{\alpha\beta}[i\delta J_\mu(y)][i\delta_k(y)]}.$$

The greens functions are the variations of $J, k, \bar{k}$, and one sets all them equal to zero.

4.3 Feynman Rules for QED

In the same fashion as section 1.10, we can summarize the last results in rules for fermions in an external gauge field. As for charged scalar fields, fermions lines carries arrows, pointing from $\bar{\psi}$ to ψ. The Feynman rules for QED are:

- Distinguishable diagrams with continuous fermions lines.
- For every fermion line, starting at z pointing to w, one adds $i(S_F(w-z))_{\alpha\beta}$, which is $i(2\pi)^{-4}\int d^4k[(k-m)^{-1}]_{ba}$.
- For every vector lines, $iG_F^{\mu\nu}(w-z)$.
- For every vertex $-ie(\gamma^\mu)_{\lambda\sigma}$, as a result of $-ieQ_f\delta_{ff'}\gamma^\mu_{dc}(2\pi)^4\delta^4(\sum_i p_i)$, where p_i labels the momenta of lines flowing.
- For each internal photon line, one adds the factor $\frac{1}{2\pi}\int d^4k\frac{1}{k^2-i\epsilon}[-g^{\mu\nu}+(1-\frac{1}{\lambda})\frac{k^\mu k^\nu}{k_i^2\epsilon}]$.
- Sign rule: (-1) for every fermion loop and (-1) when exchanging pairing of external $\psi, \bar{\psi}$.

In the momentum space, the rules are the similar with a few exceptions,

- For every fermion line, starting at z pointing to w, one adds $\int\frac{d^4k}{(2\pi)^4}\frac{i(\not{k}+m)}{k^2-m^2-i\epsilon}$.
- For every vector lines, $\int\frac{d^4k}{(2\pi)^4}\frac{i}{k+i\epsilon}\Big(-g^{\mu\nu}+(1-\frac{1}{\lambda}\frac{k^\mu k^\nu}{k^2+i\epsilon})\Big)$.
- For every vertex $-ie(2\pi)^4\delta^4(\sum_i k_i)(\gamma_\mu)_{\beta\alpha}$.

4.4 Reduction

The use the results of the Green's functions to to find the S-matrix, we need the residues of single-particle pole defined by

$$\int d^4 x e^{ipx} \langle 0 | T(\psi(x)\bar{\psi}(0)) | 0 \rangle .$$

The reduction for $e^+e^- \to e^+e^-$ in terms of the T-matrix is given by $S = 1 + iT$, where

$$iT\Big((p',\lambda')^-, (q',\sigma')^+_{out}; (p,\lambda), (q,\gamma)_{in}\Big) = \Big(\frac{\bar{u}_\alpha(p',\lambda')}{(R_\psi(2\pi)^3)^{\frac{1}{2}}}\Big)\Big(\frac{-\bar{v}_\beta(q,\sigma)}{(R_\psi(2\pi)^3)^{\frac{1}{2}}}\Big)$$

$$\times G_{\alpha'\beta',\alpha\beta}(p',q',p,q)_{trun} \times \Big(\frac{u_\alpha(p,\lambda)}{(R_\psi(2\pi)^3)^{\frac{1}{2}}}\Big)\Big(\frac{-v_\beta(q',\sigma')}{(R_\psi(2\pi)^3)^{\frac{1}{2}}}\Big)\Bigg|_{p^2=p'^2=q^2=q'2=m^2}$$

The two first parenthesis are the outgoing and incoming fermions and the two last the incoming and outgoing fermions, respectively.

For vectors, the rules are the same, to get the S-matrix: we go to the momentum space, truncate it and multiply it by, $\frac{\epsilon^\mu(k)}{(R_A(2\pi)^3)^{\frac{1}{2}}}$, the incoming vector, $\frac{\epsilon^\mu(k')^*}{(R_A(2\pi)^3)^{\frac{1}{2}}}$, the outgoing vector.

4.5 Compton Scattering

$$\gamma e^- \to \gamma e^- .$$

The S-matrix for the *Compton Scattering* is

$$\epsilon'^\nu(l') * \bar{u}_\beta(p',\sigma') G_{trun} \epsilon^\mu(l) \cdot u_\alpha(p',\sigma) . \frac{1}{R_A R_\psi (2\pi)^6} .$$

4.6 The Bhabha Scattering

The *Bhabha scattering* is an example of tree level gauge theory cross section.

$$e^+e^- \to e^+e^- .$$

The S-matrix level is gauge invariant (independent of gauge theory fixing),

$$iM(p_4,\sigma_4,p_3,\sigma_3,p_2,\sigma_2,p_1,\sigma_1) =$$

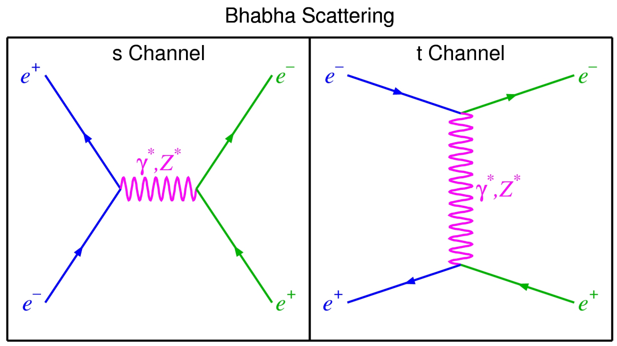

Figure 4.1: The two tree Feynman diagrams of the Bhabha Scattering. In the internal line, μ and ν couple together by the propagator of the photon. The first diagram is the scattering exchange and the second the annihilation followed by a pair creation, All momentum are on-shell.

There are two diagrams (figure 4.1),

$$
\begin{aligned}
i\mathcal{M} = (2\pi)^4\delta^4\Big(\sum_{i=1}^{4} k_i^\mu\Big) \times \Big[\Big(\frac{i}{-k_2 - m}[-ie\gamma^\mu]\frac{i}{k_4 - m}\Big)_{bd} - \\
-\frac{i}{|k_2+k_4|^2}\Bigg(- g_{\mu\nu} + (1-\frac{1}{\lambda})\frac{1}{|k_2+k_4|^2}\Bigg) \times \\
\Big(\frac{i}{-k_3 - m}[-ie\gamma^\mu]\frac{i}{k_1 - m}\Big)_{ca} \\
- \text{other diagrams where } k_2 \leftrightarrow k_3, b \leftrightarrow c.
\end{aligned}
$$

The Green's function is always pure imaginary for tree diagrams and T and $\mathcal{M}$ are always real. From the result of the reduction,

$$T = \sigma^4\Big(\sum k_i\Big)\frac{1}{(2\pi)^6}\mathcal{M},$$

where we only need to find $\mathcal{M}$. From the diagrams,

$$i\mathcal{M}(\{p_i,\sigma_i\}) = \bar{v}(p_2,\sigma_2)(-ie\gamma)^{\mu}v(p_4,\sigma_4)\times$$
$$\frac{1}{(p_2-p_4)^2}\Big(-g_{\mu\nu}+(1-\frac{1}{\lambda})\frac{(p_2-p_4)_{\mu}(p_2-p_4)_{\nu}}{(p_2-p_4)^2}\Big)\times$$
$$\bar{u}(p_2,\sigma_2)(-ie\gamma)^{\mu})u(p_1,\sigma_1)-$$
$$-\bar{v}(p_2,\sigma_2)(-ie\gamma)^{\mu})v(p_1,\sigma_1)\times$$
$$\frac{1}{(p_2-p_1)^2}\Big(-g_{\mu\nu}+(1-\frac{1}{\lambda})\frac{(p_2-p_1)_{\mu}(p_2-p_1)_{\nu}}{(p_2-p_1)^2}\Big)\times$$
$$\bar{u}(p_3,\sigma_3)(-ie\gamma^{\mu})u(p_4,\sigma_4)\Big],$$

which still depends on the gauge fixing parameter λ. The Green's function also depend on the gauge while the physical amplitudes does not. The terms multiplied by λ are dropped and one needs to keep $-\frac{g_{\mu\nu}}{k^2}$ for photon propagation. Simplifying (4.1), the first diagram is

$$=\bar{v}_2(-e\gamma^{\mu})u_1\left(\frac{-ig^{\mu\nu}}{(p_1+p_2)^2}\right)\bar{u}_3(-e\gamma^{\mu})v_4.$$

The second is

$$=-\bar{v}_2(-e\gamma^{\mu})v_4\left(\frac{-ig^{\mu\nu}}{(p_1+p_4)^2}\right)\bar{u}_3(-e\gamma^{\mu})u_1.$$

Hence, one has

$$\mathcal{M}=\bar{v}_2(-ie\gamma)^{\mu}v_4\Big(\frac{1}{(p_2-p_4)^2}\Big)\bar{u}_3(-ie\gamma)^{\mu}u_1-$$
$$\bar{v}_2(-ie\gamma)^{\mu}u_1\Big(\frac{1}{(p_1-p_3)^2}\Big)\bar{u}_3(-ie\gamma)^{\mu}v_4.$$

The gauge invariance of the general result for the S-matrix is common to all orders in QED and the other gauges theories. In general, $\mathcal{M}$ can depends on all invariants $p_ip_j, p_i\sigma_j, \sigma_j\sigma_k$, they are all Lorentz invariant number. However, certain dependences are not allowed by parity of QED. For example, let us suppose we have

$$\mathcal{M}\propto A(\{p_1,p_2\})+\sigma_ip_jB(\{p_i,p_j\}).$$

In the frame $\sigma^0=0$ this changes the spin under $xp-x$. In transformed fields where $x=-x$, $\mathcal{M}=A-\sigma_ip_jB$, not allowed in QED, however allowed in the weak interactions.

The *Feynman identity* (a generalization of the Ward identity) is

$$\bar{u}(p_2)(\not{p}_2-p_1)u(p_1)=0.$$

4.7 Cross Section

For a fixed spin, the relations for $\mathcal{M}$ are the same as for scalars. We can now compute the cross section for fermion+antifermion scattering,

$$\frac{d\sigma}{dt} = \frac{1}{16\pi s(s-4m^2)}|M(\{p_i,\sigma_i\})|^2,$$

where $t = (p_3 - p_1)^2 = (p_4 - p_2)^2, s = (p_1 + p_2)^2, u = (p_4 - p_1)^2$.

4.8 Dependence on the Spin

Most experiment involves non-polarized beams at the target, i.e. incoherent combination of spins. For unpolarized σ's, we average over initial state spin and then sum over the final state,

- First we look to the general initial state: $|A_{in}\rangle = \sum_{i=1}^{N} C_i^A|\{\sigma_i\}\rangle$, where N is the number of different spin states, for example, for two particles with spin-half, $N = 4$.

- If we have a unpolarized beam for a particle process $A \to$ B, then

$$\begin{aligned} |M_{A\to B}|^2 &\propto |\langle B_{out}|A_{in}\rangle|^2, \\ &= \sum_{u,j=1}^{N} C_j^* C_i \langle\sigma_j|B_{out}\rangle\langle B_{out}|\sigma_i\rangle. \end{aligned}$$

- If there is no polarization, we average the C's: $\langle C_i|C_j^*\rangle = \frac{1}{N}\delta_{ij}$, and one has $\langle|M_{A\to B}|^2\rangle = \frac{1}{N}\sum_i |\langle B_{out}|\sigma_i\rangle|^2$, which is the average over the initial states.

- For the final states, just sum over B's definitive spin. Applying to Bhabha scattering, we then see the simplification in the unpolarized case (it is not necessary to compute $\mathcal{M}$ for the fixed spin).

- Rather than calculate $\bar{v}_2\gamma^* v_1$, etc, we use the complex conjugate of the identity $\gamma^0\gamma^\mu\gamma = (\gamma^\mu)^\dagger$. Hence, any factor $(\bar{u}_i, \gamma_{\alpha 1}, \gamma_{\alpha 2}, ...\bar{\gamma}_{\alpha n}, u_j)^* = \bar{u}_j\gamma_{2n}\gamma_{2n-1}...\gamma_1 u_i$. This will generate four therms in the Bhabha scattering, which are the four cut diagrams, related to the optical theorem.

Calculating for the first cut diagram, we have

$$\frac{e^4}{s^2}\sum_s\{(\bar{v}_2\gamma^\mu u_1)(\bar{u}_1\gamma^\nu v_2)(\bar{u}_3\gamma_5 v_4)(\bar{v}_4\gamma^\mu u_3)\}.$$

Using the projection relations,

$$\sum_s(u_i)_\beta(\bar{u}_i)_\alpha = \sum_\lambda \bar{u}_\beta(p_i\lambda)u_\alpha(p_i\lambda) = (\not{p}_1 + m)_{\alpha\beta},$$
$$\sum_s(v_i)_\beta(\bar{v}_i)_\alpha = \sum_\lambda \bar{v}_\beta(p_k,\lambda)v_\lambda(p_j,\lambda) = (\not{p}_2 + m)_{\alpha\beta},$$

finally giving

$$\frac{e^4}{s^2}\sum_s\Big\{\text{ Tr }_{dirac}[(\not{p}_1+m)\gamma^\nu(\not{p}_2+m)\gamma^a].\text{ Tr }_{dirac}[(\not{p}_3+m)\gamma^\mu(\not{p}_4+m)\gamma^{nu}]\Big\}.$$

The following other three diagrams have: (b)-2 traces and (c), (d)-one trace. For the second of the cut diagrams, we have

$$\frac{e^4}{s^2}\sum_s\{\bar{u}_\beta(p_3,\lambda_3)(\gamma^\mu)_{\beta\gamma}u_\gamma(p_1,\lambda)\bar{u}_\lambda(p_1,\lambda_1)(\lambda^\tau)_{\gamma\tau}u_\tau(p_3,\lambda_3)\},$$

and the only trace is given by the piece in the middle,

$$u_\lambda(p_1,\lambda)\bar{u}_\lambda(p_1,\lambda_1) = (\not{p}_1 + m)_{\lambda\sigma}.$$

It is straightforward to find the other two cut diagrams.

4.9 Diracology and Evaluation of the Trace

To calculate the traces of the diagrams in the Feynman, we define useful proprieties of the Dirac matrices,

$$\begin{aligned}
\text{tr } \{\not{a}\not{b}\} &= 2ab, \\
\text{tr } \gamma^{\mu} &= 0, \\
\text{tr } \ (\text{odd numbers of } \gamma) &= 0, \\
\text{tr } [\gamma_5 \not{a}\not{b}\not{c}\not{d}] &= 4i\epsilon_{\mu\nu\lambda\nu} a^{\mu} b^{\nu} c^{\lambda} d^{\sigma}, \\
\text{tr } [\not{a}\not{b}\not{c}\not{d}] &= 4(abcd + adbc - acbd), \\
\text{tr } [\sum_i \not{a}_i] &= \sum_{i=1}^{n-1} a_n a_i^(- 1)^{i-1} \text{ Tr} \\
\gamma_5 &= i\gamma_0\gamma_1\gamma_2\gamma_3, \\
(\gamma_5)^2 &= 1, \\
(\gamma^0)^{\dagger} &= \gamma^0, \\
(\gamma^{\mu})^{\dagger} &= \gamma^0\gamma^{\mu}\gamma^0, \\
\{\gamma_5, \gamma_{\mu}\} &= 0, \\
\not{a}\not{b} &= ab - 2ia_{\mu} S^{\mu\nu} b_{\nu}, \\
S^{\mu\nu} &= \frac{i}{4}[\gamma^{\mu}, \gamma^{\nu}], \\
\gamma_{\mu}\not{a}\gamma^{\mu} &= -2\not{a}, \\
\gamma_{\mu}\not{a}\not{b}\gamma^{\mu} &= 4ab, \\
\gamma_{\mu}\not{a}\not{b}\not{c}\gamma^{\mu} &= -\gamma_{\mu}\not{a}\not{b}\gamma^{\mu}\not{c} + 2\not{c}\not{a}\not{b}, \\
&= -s\not{c}\{\not{a}, \not{b}\} + 2\not{c}\not{a}\not{b}, \\
&= -2\not{a}\not{b}\not{c}, \\
\{\not{a}, \not{b}\} &= 2ab.
\end{aligned}$$

Now, we estabilish the follow relations,

$$
\begin{aligned}
&\text{tr} \quad M_{ab} = M_a^a, \\
&\text{tr} \quad \gamma_\mu = 0, \\
&\text{tr} \quad \gamma_5 = 0, \\
&\text{tr} \quad \gamma_\mu\gamma_\nu\gamma_\lambda = 4g_{\mu\nu}, \\
&\text{tr} \quad \gamma_\mu\gamma_\nu = 0, \\
&\text{tr} \quad \not{a}\not{b} = 4g_{\mu\nu}, \\
&\text{tr} \quad \gamma_5\gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma = 4i\epsilon_{\mu\nu\lambda\sigma}, \\
&\text{tr} \quad \gamma_\mu\gamma_\nu\gamma_\lambda\gamma_\sigma = 4(\eta^{\mu\nu}\eta^{\rho\sigma}\eta^{\rho\sigma} - \eta^{\mu\rho}\eta^{\nu\gamma} + \eta^{\mu\sigma}\eta^{\nu\rho}), \\
&\text{tr} \quad (\not{a}_1\not{a}_2\not{a}_3\not{a}_4) = 4[(a_1a_2)(a_3a_4) - (a_1a_3)(a_2a_4) + (a_1a_4)(a_2a_3)], \\
&\text{tr} \quad (\not{a}_1\not{a}_2 ... \not{a}_n) = \sum_{i=1}^{n-1}(a_n.a_i)(-1)^{i-1}\text{tr } (\not{a}_1\not{a}_2 ... \not{a}_{n-1}).
\end{aligned}
$$

Finally, we evaluate the traces,

$$
\begin{aligned}
t^{b_1}_{\mu\nu} &= \text{Tr } [\gamma^\nu(\not{p}_1 + m)\gamma^\mu(\not{p}_3 + m)], \\
&= \text{Tr } [\gamma^\nu\not{p}_1\gamma^\mu\not{p}_3] + m^2 \text{ Tr } [\gamma^\nu\gamma^\mu], \\
&= 4(p_1^\nu p_3^\mu + p_3^\nu p_1^\mu - g^{\nu\mu}p_1p_3) + 4m^2p_1p_3.
\end{aligned}
$$

and

$$
\begin{aligned}
t^{b_2}_{\mu\nu} &= \text{Tr } [\gamma^\nu(\not{p}_2 + m)\gamma^\mu(\not{p}_4 + m)], \\
&= \text{Tr } [\gamma^\nu\not{p}_2\gamma^\mu\not{p}_4] + m^2 \text{ Tr } [\gamma^\nu\gamma^\mu], \\
&= 4(p_2^\nu p_4^\mu + p_4^\nu p_2^\mu - g^{\nu\mu}p_2p_4) + 4m^2p_2p_4.
\end{aligned}
$$

The full trace results to be a Lorentz invariant,

$$
\begin{aligned}
T_b &= t^{b_1}_{\mu\nu}(t^{b_2})^{\mu\nu} \\
&= 16(p_1^\nu p_3^\mu + p_3^\nu p_1^\mu - g^{\mu\nu}p_1p_3)(p_{2\nu}p_{4\mu} + p_{4\nu}p_{2\mu} - g^{\mu\nu}p_2p_4) - 2m^2(p_1p_3 + p_1p_3).
\end{aligned}
$$

Finally, we are in the very moment we can calculate the cross section,

$$
\begin{aligned}
\frac{\sigma}{dt} = \frac{2\pi\alpha^2}{s(s^2-4m)}\{ \quad & \tfrac{1}{s2} \quad [(u-2m^2)^2 + (t-2m^2)^2 + 4m^2s] \\
+ \quad & \tfrac{1}{t^2} \quad [(u-2m^2)^2 + (s-2m^2)^2 + 4m^2s] \\
+ \quad & \tfrac{1}{st} \quad [(u-2m^2)^2 + 4m^2(u-2m^2)] \ \},
\end{aligned}
$$

where $\alpha = \frac{e^2}{4\pi}$ and the Mandelstam variables are

$$\begin{aligned} s &= (p_2 + p_2)^2, \\ t &= (p_1 - p_3)^2 = (p_2 - p_4)^2, \\ u &= (p_1 - p_4)^2 = (p_2 - p_3)^2. \end{aligned}$$

The behavior of the cross section in terms of the center of mass scattering angle is derived from (4.9.1) for the spin average case with equal mass particles. Let θ be the center of mass angle between p_i and p_3, we rewrite the Mandelstam variables as

$$\begin{aligned} s &= 4E^2, \\ t &= -4(E^2 - m^2)\sin^2\frac{\theta}{2}, \\ u &= -4(E^2 - m^2)\cos^2\frac{\theta}{2}. \end{aligned}$$

The previous calculation is simplified when we take the limit $m \to 0$, which is the same limit which E and the*moment transfered* go to zero (for a fixed m). There is no specified spin dependence but the answer depends on the spin of the field.

Chapter 5

Electroweak Theory

The electroweak theory is represented by the Lagrangian

$$\mathcal{L}_{EW} = - \frac{g}{\sqrt{2}} \sum_{charged} \Big[\bar{u}_1^{(1)} \not{W}^{+} l_i^{(L)} + \bar{l}_i \not{W}^{-} u_i^{(L)} \Big]$$

$$- \frac{g}{cos\theta_W} \sum_{neutral} \Big[\bar{u}_1^{(1)} \not{Z} u_i^{(L)} + \bar{l}_i^{(L)} \not{Z} (\gamma_5 + 4sin^2\theta_W - 1) \Big].$$

It consists of two parts, a *charged current and interactions*, coupling charged leptons to neutrinos (only left-handed), and a]it neutral current, coupling Z boson to neutrinos or charged leptons (left-handed and right-handed). The relevant vertice are

$$-\frac{ig}{2\sqrt{2}} [\gamma^{\mu}(1-\gamma_5)]_{\beta\alpha},$$

$$-\frac{ig}{2cos\theta_W} [\gamma^{\mu}(1-\gamma_5)]_{\beta\alpha},$$

$$-\frac{ig}{4cos\theta_W} [\gamma^{\mu}(\gamma_5 + (4sin^2\theta_N - 1))]_{\beta\alpha}.$$

The propagator is

$$\Big(\frac{i}{\not{k} - m}\Big)_{\alpha\beta} \frac{i}{q^2 - M^2} \Big(- g_{\mu\nu} + \frac{q_{\mu} q_{\nu}}{M^2} \Big).$$

5.1 The Standard Model

The *Standard Model of Particles* is represented by three gauge field theories, $SU(3) \times SU(2) \times U(1)$. The group and fields couplings are described in the table 5.1.

SU(3)	SU(2)	U(1)
G_a^μ, a = 1, ..., 8	B_b^μ,$b = 1, 2, 3.$	C^μ
G are gluons	B is $W^\pm$ and Z	C is γ
g_s	g	g'

Table 5.1: The Standard Model.

For each Dirac and scalar field, one must still specify how they couple. $U(1)$ Is related to the charge, generates the phase transformation on the hypercharge Y ($e^{i\frac{Y}{2}\alpha(x)}$). $SU(2)$ is related to the electroweak interactions, giving by the $Z, W^\pm$ bosons.

The Lagrangian of the Standard Model is

$$\mathcal{L}_{\mathcal{SM}} = \mathcal{L}_{vectors} + \mathcal{L}_{higgs} + \mathcal{L}_{quarks} + \mathcal{L}_{leptons},$$

$$\mathcal{L}_{vectors} = -\frac{1}{2}\ \mathrm{Tr}_{SU(3)}[F_{\mu\nu}^2(A)] - \frac{1}{2}\ \mathrm{Tr}_{SU(2)}[F_{\mu\nu}^2(B)] - \frac{1}{4}F_{\mu\nu}^2(C).$$

where the two first terms are bosons and the two last fermions.

The general form of the covariant derivative is

$$D_\mu[A, B, C] = \partial_\mu + ig_s A_{\mu,a} T_a^{(R_f^3)} + igB_{\mu,b} T_b^{(R_e^2)} + ig'\frac{Y}{2}C_\mu.$$

The Standard Model distinguishes left-handed and right-hand parts of the Dirac field, where

$$\psi_L = \frac{1}{2}(I \mp \gamma_5)\psi.$$

Singlets fields (under $SU(3), SU(2), U(1)$) have a missing term D^α. For example, leptons are singlets under $SU(3)$, and other cases can be seen in the table 5.2.

	$R_f^{(3)}$	$R_e^{(2)}$	Y
q_R^k	3	1,1	4/3, -2/3
q_L^k	3	2 (u and d)	1/3
l_R^k	1	1,1	0,-2
l_L^k	1	2 (ν_e and e)	-1
ϕ	1	2	1

Table 5.2: Summary of the representations of the Standard Model.

Hypercharge Construction

The way we represent the hypercharge in the gauge group formalism is by writing

$$Q_{EM} = \frac{Y}{2} + I_3^{SU(2)_L},$$

where I_3 is equal to 1/2 for U_l, μ_l and -1/2 for d_l, l. The symmetry $SU(2)$ does not allow a Dirac mass term: $m\bar{\psi}\psi = m(\bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L)$, since ψ_L and ψ_R transforms differently. These fields gets mass by the same Higgs mechanism of vectors. The Higgs fields are represented as

$$\Phi = \begin{pmatrix} \phi^\dagger, I_3 = \frac{1}{2} \\ \phi^0, I_3 = -\frac{1}{2} \end{pmatrix}.$$

Chapter 6

Quantum Chromodynamics

6.1 First Corrections given by QCD

The Lagrangian of QCD is

$$\mathcal{L} = \bar{\psi}_i(i\gamma^\mu \delta_\mu - m)\psi_i - g(\bar{\psi}_i \gamma^\mu T^a_{ij}\psi_i)G^a_\mu + \frac{1}{4}G^a_{\mu\nu}G^{\mu\nu}_a. \tag{6.1.1}$$

In this section, we will calculate the first corrections due the strong force: the first-order radiation of gluons jets on the process $e^+e^- \rightarrow g\bar{q}q$, a QCD's process[1]. This can be seen as an coupling terming of fermions to gluons at the Lagrangian,

$$g\bar{\psi}_{fi}\gamma^\mu \gamma_{fi} B_\mu, \tag{6.1.2}$$

with color indexes $i = 1, 2, 3$ and f the quark flavors. One can then calculate the cross section for emission of one gluon. Writing the four-momentum of the equation as $q^\mu = q_0, \vec{q}$, and the momentum of quark, anti-quark and gluon as $k^\mu_i - (\omega_i, \vec{k})$, i = 1,2 and 3, the ratio of the center-of-mass energy of the particle i to the maximum available energy is given by

$$x_i = \frac{2k_i q}{q^2} = \frac{2E_i}{\sqrt{s}}, \tag{6.1.3}$$

where $\sum x_i = 2$. To write the differential cross section, equation 6.1.4, one needs to calculate the phase space and matrix $|\mathcal{M}|^2$. From 6.1.3 one can calculate the range of integration, finding $0 \leq x_q, x_2 \leq 1 - \frac{\mu}{q}$.

$$d\sigma_n = \frac{1}{q^2}[\frac{1}{4}\sum_{spins}|\mathcal{M}|^2][\int \Pi_n]. \quad (6.1.4)$$

Choosing the frame where $\vec{q}=0$, one has $k_1+k_2+k_3=0$, the three-body phase space is given by

$$\begin{aligned}
\int \Pi_3 &= \int \frac{d^3k_1 d^3k_2}{(2\pi)^5 8\omega_1\omega_2\omega_3}\delta(q_0-\omega_1-\omega_2-\omega_3), && (6.1.5)\\
&= \frac{1}{32\pi^3}\int d\omega_1 d\omega_2 \sigma(q_0-\omega_1-\omega_2-\omega_3), && (6.1.6)\\
&= \frac{q^2}{128\pi^3}\int dx_1 dx_2. && (6.1.7)
\end{aligned}$$

Calculating explicitly the two Feynman diagrams with emission of one gluon, squaring $\mathcal{M}$, finding the two traces, electron (pure QED) and quarks (some extra tricks, see [PS1995]), one has

$$\begin{aligned}
\frac{1}{4}\sum_{spins}|\mathcal{M}|^2 &= \frac{1}{4}\Big(\frac{e^4g^2Q_F^2}{q^4}\Big)\ \mathrm{Tr}\ [\not{p}\gamma^\mu\not{p}\gamma^\alpha]\ \mathrm{Tr}\ [\not{p}\Lambda_{\beta\mu}\not{p}\Lambda_{\alpha\beta}], && (6.1.8)\\
&= \frac{e^4g^2Q_F^2}{4q^4} I^{\mu\alpha}G_{\mu\alpha}. && (6.1.9)
\end{aligned}$$

Gathering everything on 6.1.4, one has the differential cross section:

$$\begin{aligned}
d\sigma &= \frac{e^4g^2Q_F^2}{4q^4}\frac{1}{256\pi}I^{\mu\alpha}\int dx_1dx_2G_{\mu\alpha}, && (6.1.10)\\
&= \frac{2\alpha^2\alpha_s^2}{3}\frac{Q_f^2}{q^2}\int \frac{8(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}dx_1dx_2. && (6.1.11)\\
\frac{d^2\sigma}{dx_1dx_2} &= \frac{8\sigma^2\sigma_s^2Q_f^2}{3q^2}\frac{8(x_1^2+x_2^2)}{(1-x_1)(1-x_2)}. && (6.1.12)
\end{aligned}$$

Integrating this on the range $0 \leq x_1, x_2 \leq 1$ gives an integral that diverges. One then sets the range of integrations are the infrared divergences as $\mu \to 0$, and integrating with $\mu \neq 0$ one gets the infrared terms proportional to $\ln(\frac{\mu}{q})$, which are related to the infrared cut of QCD. These singularities are not physical but the breakdown of perturbation theory, meaning that the mass of quarks and gluons are never on-shell, as was supposed on these calculations (e.g. ω_i cannot be of the same order of the hadron masses).The 2-3 kinematics collapses to an effective 2-2 due to an emission of soft gluon or to collinear splitting of a parton on two. There are then two possible emissions, soft (when $\frac{\omega_3}{\sqrt{s}} \to 0$) and collinear (when $\theta_{i3}, (1-x_i) \to 0$ from $(1-x_i) = \frac{x_j \omega_3}{q}(1 - \cos\,\theta_{j3})$).

6.2 The Gross-Neveu Model

The Gross-Neveu model is a quantum field theory model of massless Dirac fermions with one spatial and one time dimension, with an attractive short-range potential, where fermion pairs composite condensates to break Z_2-symmetry and acquire mass. It was introduced as a toy model for quantum chromodynamics.The Lagrangian of the model is given by

$$\mathcal{L} = \bar{\psi}_i i \partial\!\!\!/ \psi_i + \frac{1}{2} g^2 (\bar{\psi}_i \psi_i)^2.$$

This theory is invariant under $\psi_i \to \gamma^5 \psi_i$ and the chiral symmetry forbids the appearance of fermions mass.

Proof. $\bar{\psi}_i \psi_i = \psi_i^\dagger \gamma^0 \psi_i = \psi^\dagger (\gamma^5 \gamma^0 \gamma^5) \psi_i = -\psi^\dagger \gamma^0 \psi_i = -\bar{\psi}_i \psi_i$. Therefore we see that $\bar{\psi}_i \psi_i = \text{const}$. Same proof can be made for $\bar{\psi}_i i \partial\!\!\!/ \psi_i$. This is a Z_2 symmetry. □

The theory is renormalizable in two dimensions.

Proof. The mass dimension in a theory in D-dimensions is $\frac{D-1}{2}$. The space-time volume has dimension $-D$. The Lagrangian density has mass dimension D. Therefore, for $D = 2$, we have $\frac{D-1}{2} = \frac{1}{2}(-\epsilon)$. Since the four-fermion operator has dimension=2, g is dimensionless and renormalizable. □

6.3 The Parton Model

The intrinsic scheme of the elementary particles in nature is given by their composed quarks. We call *parton* all the quarks and gluons of inside a particle. We represent the parton distribution in the *vacuum sea* by

$$u_s(x) = \bar{u}_s(x) = d_s(x) = \bar{d}_s(x) = s_s(x) = \bar{s}_s(x) = s(x). \tag{6.3.1}$$

On another hand, we represent the *valence quarks*, such as in the proton,

$$u(x) = u_v(x) + u_s(x), \tag{6.3.2}$$

$$d(x) = d_v(x) + d_s(x). \tag{6.3.3}$$

Summing all the contribution of these partons, it is necessary to recover charges, baryon number and strangeness,

$$\int_0^1 [u(x) - \bar{u}(x)]dx = 2, \tag{6.3.4}$$

$$\int_0^1 [udx) - \bar{d}(x)]dx = 1, \tag{6.3.5}$$

$$\int_0^1 [s(x) - \bar{s}(x)]dx = 0. \tag{6.3.6}$$

The presence of gluons emissions is signaled by a quark jet and a gluon jet in the final state, either in the direction of the virtual photon ($p_T \neq 0$).

Deep Inelastic Scattering

At a very high resolution (when the transfered momentum q^2 is large), the nucleon can be resolved as a collection of almost non-interacting point-like constituents, the partons. When the resolution scale λ of the effective probe q is smaller than the typical size of the proton (~ 1 fm), the internal structure of the proton is probed and we have the *deep inelastic regime* (DIS), 6.1-a. Defining $Q^2 = -q^2 > 0$ as the off-shell momentum of the exchanged photon, the resolution scale is $\lambda = \frac{\hbar}{Q}$.

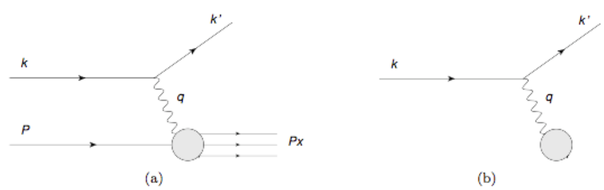

Figure 6.1: a)DIS, b) Elastic scattering.

The electron-quark scattering, $e^-(k)+q(p_q) \to e^-(k')+q(p'_q)$, is given by the QED differential cross section (partonic massless limit ($\hat{s}+\hat{t}+\hat{u}=0$)), where the matrix element squared for the amplitudes are

$$\sum |\mathcal{M}|^2 = 2e_q^2 e^4 \frac{\hat{s}^2 + \hat{u}^2}{\hat{t}^2}.$$

In terms of the Mandelstam variables, $\hat{s} = (k+p_q)^2, \hat{t} = (k-k')^2, \hat{u} = (p_q - k')^2$. In the frame which the proton is moving very past, $P \gg M$, we can consider a simple model where the photon scatter a pointlike quark with ϵ fraction of the momentum vector $p = \xi P$. The deep inelastic kinematics are $\hat{t} = q^2, \hat{u} = \hat{s}(y-1), \hat{s} = \xi Q^2/xy$. The massless differential cross section is then

$$\frac{d\hat{\sigma}}{d\hat{t}} = \frac{1}{16\pi\hat{s}^2}\sum|\mathcal{M}|^2,$$

where, substituting the kinetic variables,

$$\frac{d\hat{\sigma}}{dQ^2} = \frac{2\pi\alpha^2 e_q^2}{Q^4}[1+(1-y)^2].$$

The mass-shell constraint for the outgoing quark $p_q'^2 = 0$ suggest that the structure function probes the quark with $\xi = x$. With s the square of the the center-of mass energy of the electron-hadron, the invariant $\hat{s}$ is

$$\hat{s} = (p+k)^2 \sim 2p\ .\ k \sim xs.$$

Writing $\int_0^1 dx\delta(x-\xi) = 1$ we have the double cross section for the quark scattering process,

$$\frac{d^2\hat{\sigma}}{dxdQ^2} = \frac{4\pi\alpha^2}{Q^4}[1+(1-y)^2]\frac{1}{2}e_q^2\delta(x-\xi).$$

The event distribution is in the $x - Q^2$ plane. The parton distribution functions (PDFs) parametrizes the structure target as 'seen' by the virtual photon and which are not computable from first principles through perturbative calculations.

The *Bjorken limit* is definite as $Q^2, q \to \infty$, with x fixed. In this limit the structure functions obey an approximate *scaling* law, i.e. they only depend on x, not in Q^2, as we can see in figure 6.2. Bjorken scaling implies that the virtual photon scatters off pointlike constituents, since otherwise the structure functions would depend on the ratio Q/Q_0, some length scale.

Dividing it by $(1+(\frac{Q^2}{xs})^2)/Q^4$, we remove the dependence of the QED cross section and the result is independent of Q^2. The result is that the structure of the proton scales to an electromagnetic probe no matter how hard it is probed.

We see an anti-screening effect where the coupling constant appears strong at small momenta, behaviors called *asymptotic freedom* and belonging to a non-abelian gauge theory. These are the only field theory with asymptotic freedom behavior with interacting vector bosons which could bind the quarks. This gauge theory confirmed the Bjorken scaling, with the evolution of the coupling very slow, following a logarithmic distribution in momentum. The scaling violations corresponds to a slow evolution of the parton distributions $\mathcal{F}_i(x)$ over a logarithmic scale Q^2.

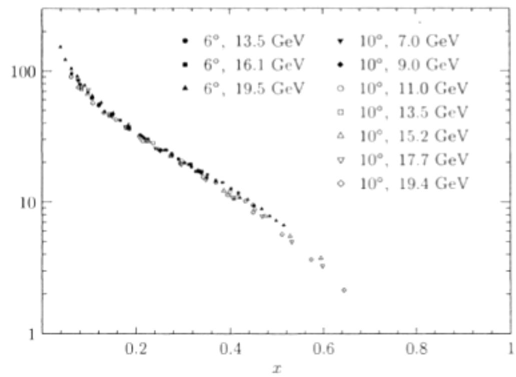

Figure 6.2: Bjorken scaling on $e^{-}p$ DIS by the SLAC-MIT experiment, range 1 GeV^2 <Q^2 <8 GeV^2 .

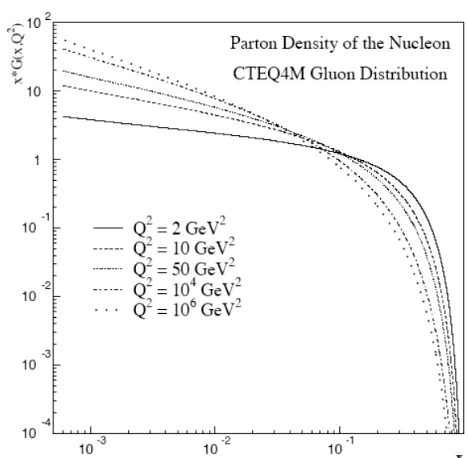

Figure 6.3: Gluon PDF. The Q^2 dependence is moderate except for very small x and/or very small Q^2.

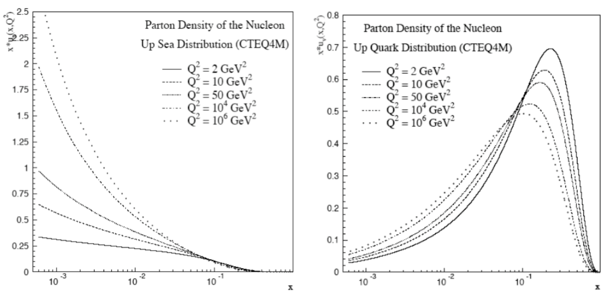

Figure 6.4: The up PDF from valence and sea. The Q^2 dependence is large for small x and small Q^2.

The Running Coupling

In the previous simple parton model, the structure functions scale in the asymptotic (Bjorken) limit $Q^2 \to \infty$. In QCD, next to the leading order in α_s, this scaling is broken by logarithms of Q^2. A quark can emit a gluon and acquire large k_T with probability proportional to $\alpha_s \frac{dk_T^2}{k_T^2}$. The integral extend up the kinetic limit $k_T^2 \sim Q^2$ and gives contributions proportional to $\alpha_s Q^2$, breaking the scaling.

The screening behaviors on QED is giving by summing the higher order corrections in terms of the general form $\alpha^n [\log(Q^2/Q_0^2)]^m$ and retaining only the leading logarithm terms ($m = n$). Therefore the vacuum polarization affects the coupling in QED as

$$\alpha(Q^2) = \frac{\alpha(Q_0^2)}{1 - \frac{\alpha(Q_0^2)}{3\pi} \log\left(\frac{Q^2}{Q_0}\right)},$$

the *leading logarithm approximation.*

For QCD, since gluons also emit additional gluons and similar charges attract themselves, the effect is anti-screening, the effective color charge decreases as one probes the original quark. Therefore, this phenomena of asymptotic freedom is given by the QCD running coupling,

$$\alpha_s(Q^2) = \frac{\alpha_s(\mu^2)}{1 + \frac{\alpha_s(\mu^2)}{12\pi}(11N_c - 2N_f) \log\left(\frac{Q^2}{\mu^2}\right)}$$

where N_c is the number of color charges, N_f the number of quarks flavors and μ the renormalization scale. To make α_s independent of the renormalization scheme, we introduce a scale Λ^2_{QCD},

$$\Lambda^2_{QCD} = \mu^2 e^{\overline{\frac{-12\pi}{(11N_c - 2N_f)\alpha_s(\mu)}}},$$

which experimentally is ~ 200. In this QCD scale, the coupling constant can be written as

$$\alpha_s(Q^2) = \frac{12\pi}{(11N_c - 2N_f)\log(Q^2/\Lambda^2_{QCD})},$$

where we see $\alpha_s \to 0$ when $Q^2 \to \infty$ and when $Q^2 \gg \Lambda^2_{QCD}$ we have hard interactions.

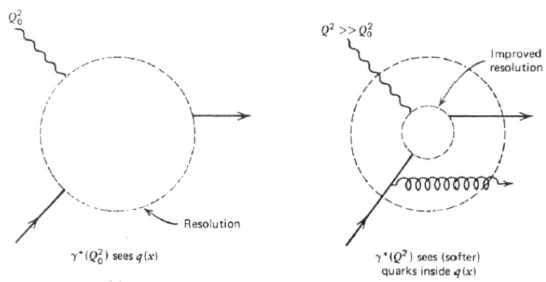

Figure 6.5: At Q^2 increases, it is possible to probe shorter distances. At large Q^2, the large x quarks are more likely to loose energy due to gluon radiation.

Leading Log Approximation

Collinear photon emission in QED at high energies is already associated with mass singularities and it leads to an analog of parton distribution for electron. Gribov and Lipatov showed that in a field theory with dimensionless coupling α, the DIS structure functions are represented as a sum of Rutherford cross sections og lepton scattering the point-like charges particle, weighted by a parton density $\mathcal{F}_i^f(x)$.

Logarithmic deviations from the true scaling behavior had been predicted for the PDFs, reveling the internal structure of PDFs which corrections are given by $\mathcal{F}_i^f(x, \log Q^2)$. Physically: DIS with a photon with Q^2 correspond to the scattering of that virtual photon on a quark of size $1/Q$. From QCD bremsstrahlung, we have

$$d\omega \propto \alpha^2 \int^{Q^2} \frac{dk_\perp^2}{k_\perp^2},$$

therefore the collinear photon emission cost a factor not of α but α log (s/m^2), and multiple collinear photons gives contributions of order $(\alpha\ \log(s/m^2))^m$. therefore it makes the total probability of extra parton production large, since for $\alpha^2 \ll 1$, log Q^2 is high enough and $\omega \propto \alpha^2$ log $Q^2 \sim 1$. In QCD, the corresponding factor for collinear gluon emission is $\alpha_s(Q^2)$ log $\frac{Q^2}{\mu^2}$.

In QED, when $k^2 \to 0$, we can write the expansion for $g^{\mu\nu}$ in terms of massless polarization vector and when the singular term as the photon momentum q goes on-shel is given by $\frac{-ig^{\mu\nu}}{q^2} \to \frac{+i}{q^2}\sum_i \epsilon^{\mu}_{Ti}\epsilon^{\nu}_{Ti}$ in the calculus of the amplitudes, decoupling the photon/electron emission vertex. The final cross section is giving by

$$\begin{aligned} \sigma(e^- X \to Y) &= \int_0^1 \mathrm{dz}\ \frac{\alpha}{2\pi}\ \log\frac{s}{m^2}\Big[\frac{1+(1-z)^2}{z}\Big] \\ &\times\ \sigma(\gamma Y \to Y), \\ &= \int_0^1 \mathrm{dz}\ \mathcal{F}_\gamma(z) \times \sigma(\gamma X \to Y). \end{aligned}$$

Considering the limit of the divergence $z = 0 = 1 - x$, the soft parton emission or infrared divergent, we see it is balanced by negative contributions from diagrams with soft virtual photons. Thus, to order α, the parton distribution of electrons has the form

$$\mathcal{F}_\gamma = \delta(1-x) + \frac{\alpha}{2\pi}\ \log\ \frac{s}{m^2}\Big(\frac{1+x^2}{1-x} - A\delta(1-x)\Big), \qquad (6.3.7)$$

where the first delta is the zeroth order of the expansion and A we will explicitly calculate in the following sections.

6.4 The DGLAP Evolution Equations

The hadron evolve with energy $Y = \log(1/x)$, and for some values of (Q^2, x), although the hadron is no longer perturbative, the evolution still is (except for low Q^2). Therefore, we can construct evolution equations in this two the two variables of the phase space: $\log(1/x)$ and $\log Q^2$.

In figure 6.6 we see that when Q^2, Y small, the proton is represented by three valence quarks and the gluon vacuum oscillations is very quickly. Increasing Q^2, the probe resolution, means to decrease the time of interaction and the probes resolves more and more of these fast fluctuations increases the number of partons seen. However, the space occupied by them decreases: at each Q^2 step the proton becomes more dilute. This evolution is described in QCD by the DGLAP equations.

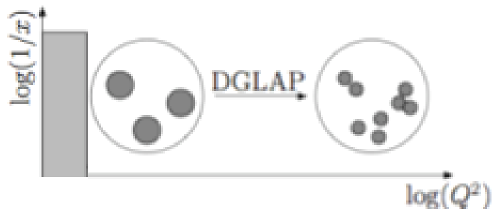

Figure 6.6: Proton in the phase space (Q^2, x).

The interaction described by small s is given by the partonic saturation, where the scales which separates the dilute regime from the dense is called saturation scale, $Q_s(x)$. For a spatial resolution greater than $1/Q_s$, gluons overlap the transverse plane, as described by the *Color Glass Condensate* formalism.

Perturbative Expansion of α_s

For a general quantity B, the perturbative expansion in terms of α_s is

$$B = \beta_0 + \beta_1\alpha_s + \beta_2\alpha_s^2 + \beta_3\alpha_s^3 ... \qquad (6.4.1)$$

where the first power of α_s is the leading order (LO), the second the next to leading order (NLO), the third the next-to-next leading order (NNLO), etc. When $Q^2 \sim \Lambda^2_{QCD}$, we have soft processes and $\alpha_s \to 1$, the expansion does not converge.

Gluon Spitting Function

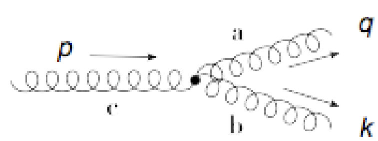

Figure 6.7: Three-gluon vertex.

The element matrix (Feynman rules) for 6.7 is

$$\begin{aligned}\mathcal{M} &= -igf^{abc}[\epsilon^*(q)\epsilon(p)\epsilon^*(k)(p+q) + \\ &+ \epsilon^*(q)\epsilon^*(k)\epsilon(p)(-q+k) - \\ &- \epsilon^*(k)\epsilon(p)\epsilon^*(q)(k+p).\end{aligned}$$

The color factor of the squared average amplitude

$$\frac{1}{d_a}\sum_{b,c} f^{abc}f^{*abc} = C_A = 3,$$

where C_A is the Casimir for the adjoint representation of SU(3) (gluon). We choose the gluon (left, right) polarization

$$\epsilon_\perp^R = \frac{1}{\sqrt{2}}(-1,-i), \epsilon_\perp^L = \frac{1}{\sqrt{2}}(1,-i).$$

The squared amplitude is then

$$\mathcal{M}^2 = \frac{12g^2p_\perp^2}{z(1-z)}\frac{z^2(1-(1-z)z)+(1-z)^2(z+(1-z)^2}{z(1-z)}.$$

In the light-cone basis, the kinematics of a a real gluon can be written as

$$\begin{aligned}p &= (\sqrt{2}p, 0.0), \\ q &\sim (\sqrt{2}zp - \frac{\sqrt{2}p_\perp^2}{4zp}, \frac{\sqrt{2}p_\perp^2}{4zp}, p_\perp, 0), \\ k &\sim (\sqrt{2}(1-z)p + \frac{\sqrt{2}p_\perp^2}{4zp}, -\frac{\sqrt{2}p_\perp^2}{4zp}, -p_\perp, 0).\end{aligned}$$

Using these kinematic relations we have

$$\begin{aligned}& \frac{1}{4\pi}\sum|\mathcal{M}|^2 \propto P_{g\leftarrow g}(z) \\ =\ & 2C_A\Big[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) \\ +\ & A\delta(1-z)\Big]\end{aligned}$$

Calculating the Normalization

We know calculate the normalization A from the last equation. We take into account the process where the gluons remains with its identity and normalize through defining a distribution that can be integrated by subtracting a delta function. We define a function that agrees to $1/(1-z)$ for all values of $x < 1$,

$$\frac{1}{(1-z)_+} = \frac{1}{(1-z)}, \forall z \in [0,1[,$$

the integral of this distribution with any smooth function $f(x)$ gives the new distribution

$$\int_0^1 dz \frac{f(z)}{(1-z)_+} = \int_0^1 dz \frac{f(z)-f(1)}{1-z}.$$

In the QED process this normalization was $A = \frac{2}{3}$. For our gluon splitting function we are going to use the relation (4.93) of [WEINBERG2005]. The leading-order DGLAP splitting function $P_{a\leftarrow b}(x)$ has an attractive interpretation as the probabilities of finding a parton of type b with a fraction x of the longitudinal momentum and a transverse momentum squared much less than μ^2. The interpretation as probabilities implies the sum rules in the leading order,

$$\int_0^1 \text{dx } P_{q\leftarrow q}(x) = 0, \tag{6.4.2}$$

$$\int_0^1 \text{dx x } \Big[P_{q\leftarrow q}(x) + P_{g\leftarrow q}(x)\Big] = 0, \tag{6.4.3}$$

$$\int_0^1 \text{dx x } \Big[2N_f P_{q\leftarrow g}(x) + P_{g\leftarrow g}(x)\Big] = 0. \tag{6.4.4}$$

We will use the last one to find the overall normalization, A, on 6.4.2, substituting the know splitting functions. For the first term,

$$\int_0^1 \text{dx x } \Big[2N_f P_{q\leftarrow g}(x)\Big] =$$
$$\int_0^1 \text{dx x } \Big[2N_f \text{ Tr } [x^2 + (1-x)^2]\Big] =$$
$$N_f \int_0^1 \text{dx x } \Big[x^2 + (1-x)^2\Big] = \frac{N_f}{3}.$$

The second therm $(x = 1 - z)$,

$$\int_0^1 \text{dx x } [P_{g\leftarrow g}(x)] =$$
$$2C_A \int_0^1 \text{dx x } \Big[\frac{x}{1-x} + \frac{1-x}{x} + x(1-x)\Big] =$$
$$6 \int_0^1 \text{dx x } \Big[\frac{x-1}{1-x} + \frac{x-1}{x} + x(1-x)\Big] =$$
$$6\Big(-\frac{11}{12}\Big).$$

Summing up both terms, we find $6A = \frac{11}{2} - \frac{N_f}{3}$. Finally, the leading-order corrected splitting function, 6.4.2, is then given by

$$P^0_{g\leftarrow g}(z) = 6\Big[\frac{z}{1-z} + \frac{1-z}{z} + z(1-z) + \Big(\frac{11}{12} - \frac{N_f}{18}\Big)\delta(1-z)\Big)\Big],$$

.

In the language of the expansion on 6.4.1, the (gluon) splittings functions can be written as

$$P_{g\leftarrow g}(z, \alpha_s) = P^0_{g\leftarrow g} + \frac{\alpha_s}{2\pi} P^1_{\leftarrow gg}(z) + ...$$

Proprieties of Splitting Functions

$$\begin{aligned} P_{q\leftarrow q}(z) &= P_{q\leftarrow q}(1-z), \\ P_{q\leftarrow g}(z) &= P_{q\leftarrow g}(1-z), \\ P_{g\leftarrow g}(z) &= P_{g\leftarrow g}(1-z). \end{aligned}$$

Because of the charge conjugation invariance and $SU(N_f)$ flavor symmetry,

$$\begin{aligned} P_{q_i\leftarrow q_j}(z) &= P_{\bar{q}_i\leftarrow \bar{q}_j} \\ P_{q_i\leftarrow \bar{q}_j}(z) &= P_{\bar{q}_i\leftarrow q_j} \\ P_{q_i\leftarrow g}(z) &= P_{\bar{q}_i\leftarrow g} = P_{q\leftarrow g} \\ P_{g\leftarrow q_i}(z) &= P_{g\leftarrow \bar{q}_i} = P_{g\leftarrow q} \end{aligned}$$

Altarelli-Parisi Equations

Together with the other splitting functions we can write explicitly the Altarelli-Parisi equations which describes the coupled evolution of the PDFs, $\mathcal{F}_g(x,Q), \mathcal{F}_f(x,Q), \mathcal{F}_{\bar{f}}(x,Q)$, for each flavor of quark and antiquarks (treated as massless at Q),

$$\frac{d}{d\log Q}\mathcal{F}_g(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\Bigg[P_{g\leftarrow q}\sum_f \Big(\mathcal{F}_f(\frac{x}{z},Q) \quad + \\ \mathcal{F}_{\bar{f}}(\frac{x}{z},Q)\Big) + P_{g\leftarrow g}(z)\mathcal{F}_g(\frac{x}{z},Q)\Bigg],$$

$$\frac{d}{d\log Q}\mathcal{F}_f(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\Bigg[P_{q\leftarrow q}\mathcal{F}_f(\frac{x}{z},Q) \quad + \\ P_{q\leftarrow g}(z)\mathcal{F}_g(\frac{x}{z},Q)\Bigg],$$

$$\frac{d}{d\log Q}\mathcal{F}_{\bar{f}}(x,Q) = \frac{\alpha_s(Q^2)}{\pi}\int_x^1 \frac{dz}{z}\Bigg[P_{q\leftarrow q}\mathcal{F}_{\bar{f}}(\frac{x}{z},Q) \quad + \\ P_{q\leftarrow g}(z)\mathcal{F}_g(\frac{x}{z},Q)\Bigg],$$

Generally, the DGLAP equation is a $(2N_f+1)$-dimensional matrix equation in the space of quarks, antiquarks and gluons.

6.5 Jets

A constituent cannot acquire a large transverse momentum except throughout exchange of gluon. A process suppressed by the smallness of α_s at large momentum scales. We have then strong gluon radiation in limit of large Q^2: if the gluon has large transverse momentum we will see three jets.

The kinematic of decay of a hadron $Q\bar{q}$ or Qqq is the kinematic of decay of Q. The four momenta is given by

$$(\sum p_i)^2 = m_Q^2, \tag{6.5.1}$$

which is difficult to measure because of the missing energy of neutrinos. A consequence of a large energy released is that the final particles will have a

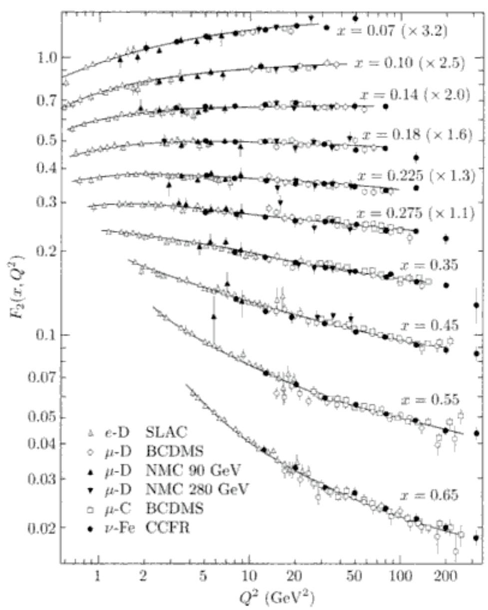

Figure 6.8: Dependence on Q^2 of the quark PDF in a DIS electron-proton. The curves are the results from Altarelli-Parisi equations.

wider solid angle when comparing with lighter quarks with same energy. For instance, when heavy quarks Q are produced near threshold $e^+e^- \to Q\bar{Q}$, they are moving slowly, their decay is isotropic. Lighter $e^+e^- \to q\bar{q}$, with same energy leads to *narrow back-to-back pairs of jets.*

6.6 Flavor Tagging

An energetic light quark q turns to a hadron forming a narrow jet collimated to original quark detection. It is not possible to determinate which flavor originated this jet. A heavy quark Q, on another hand, contains decay products that dominate its proprieties. For example a muon can be produced with a transversal momentum up to $p_T^\mu < \frac{1}{2}m_Q$.

6.7 Quark-Gluon Plasma

The initial collision in qq, qg or gg happens at ~ 0.1 fm/c and at temperatures higher than 200-300 MeV. Calculations on Lattice had a value for the critic temperature around $T_C = 170$ MeV. At this point the QGP is formed, such as in 10^{-6} seconds after the Big Bang. The whole process is described in the following:

$0.1 - 0.6$ **fm/c** System is in *local equilibrium* and it obeys hydrodynamical properties, i.e. matter has collective flow. The relation viscosity/entropy is low. The fluid does not desaccelerate. The entropy per unit of rapidity is conserved, i.e. the particle production per rapidity (entropy) does not depend on the details of the hydrodynamic evolution, but only in the initial energy (entropy).

$0.6 - 5$ **fm/c** System is in a very low viscosity state and already in QGP. The initial temperature at RHIC is about $300 - 600$ MeV.

$3.5 - 7$ **fm/c** The hadronization takes place. It is now a very dense system, with cascade of collisions and hadrons in excited state.

7 **fm/c** Freeze-out.

Signatures of QGP

Strangeness Production. In NN-collisions, the multiplicity increases faster. If strange quarks are in thermal equilibrium with lighter counterpart, the ratio of the strange to non strange has an enhancement compared to non-equilibrium situations.

Suppression of J/ψ and ψ production. Because of the color screening, the interaction between c and $\bar{c}$ is diminished by the presence of other quarks.

Two particle interferometry.

Thermal Radiation.

Dilepton Production.

Chapter 7

Renormalization

7.1 Gamma and Beta Functions

The *gamma function* is defined as

$$\Gamma(z) = \int_0^\infty dx x^{z-1},$$
$$\Re z > 0.$$

This integral does not exist for $\Re z < 0$. We have the following proprieties,

$$z\Gamma(z) = \Gamma(1+z),$$
$$\Gamma(z-1) = \frac{\Gamma(z)}{z-1},$$

where $\Gamma(z)$ has isolated points at negative integers and $z = 0$. We can show the residue of the pole at $z = -N$ is $\frac{(-1)^N}{N!}$.

The *beta function* is defined as

$$\begin{aligned}
B(\mu,\nu) &= \frac{\Gamma(\mu)\Gamma(\nu)}{\Gamma(\mu+\nu)}, \\
B(\mu,\nu) &= \int_0^1 dx x^{\mu-1}(1-x)^{y-1}, \\
&= \int_0^\infty dy y^{\nu-1}(1+y)^{-\mu-\nu}, \\
&= 2\int_0^{\pi/2} d\theta (sin\theta)^{2\mu-1}(cos\theta)^{2\nu-1}.
\end{aligned}$$

For $\Gamma(1) = 1$ we must expand it around to determine the beta function of $\Gamma(z)$ near to $z = 0, -1, -2...$ For $\epsilon \gg 1$, we have

$$\Gamma(\epsilon) = \frac{\Gamma(1+\epsilon)}{\epsilon},$$
$$\Gamma(1+\epsilon) = 1 - \epsilon\gamma_E + \epsilon^2(-\frac{\gamma_E^2}{2} + \frac{\pi^2}{12}) + ...$$
$$\Gamma(1+\epsilon) = e^{-\epsilon\gamma_E + \epsilon^2\frac{\pi^2}{12}} + \mathcal{O}(e^2),$$
$$\Gamma(N) = (N-1)!.$$

7.2 Dimension Regularization

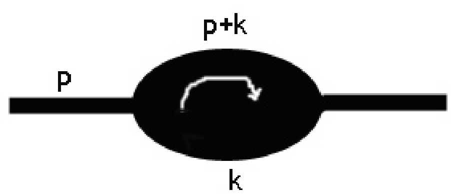

Figure 7.1: The self-integral.

The self-integral is equal to

$$= C\int \frac{d^4k}{[(p+k)^2 - m^2 + i\epsilon][k^2 - m^2 + i\epsilon]}$$

Working in N dimensions, where $N < 4$, the Feynman parameterization of the integral is giving starting with the identity,

$$\frac{1}{AB} = \int_0^1 dx \frac{1}{[xA + (1-x)B]^2},$$

where

$$\begin{aligned} I_n(p^2) &= \int \frac{d^4k}{[(p+k)^2 - m^2 + i\epsilon]} \frac{1}{[k^2 - m^2 + i\epsilon]}, \\ &= \int d^nk \int_0^1 \frac{dk}{[x(p+k)^2 - xm^2 + (1-x)k^2 - (i-x)m^2 + i\epsilon[x + (1-x)]]^2}, \\ &= \int_0^1 \int d^nk \frac{1}{[k^2 + 2pxk + xp^2 - m^2 + i\epsilon]^2}, \\ &= \int_0^1 dx \int d^{n-1}\vec{l} \int \frac{dl_0}{[l_0^2 - l^2 + x(1-x)p^2 - m^2 + i\epsilon]^2}, \end{aligned}$$

in the last, the square was completed.

Poles are at $l_0 = \pm\sqrt{l^2 - x(1-x)p^2 + m^2 - i\epsilon}$, as long as $p^2 < 4m^2$. We then choose to perform the integral by Wick rotation and the original contour turns to a closed contour with no poles (-$i\infty$ to $i\infty$). From Cauchy,

$$\int_{-\infty C_1}^{\infty} dl_0 = \int_{-\infty C_2}^{\infty} dl_0,$$

$$l\eta = il_0,$$

$$\int_{-\infty}^{\infty} dl_0 = i \int_{-\infty}^{\infty} dl\eta.$$

Along C_2 we get

$$I_\eta(p^2) = i \int_0^1 dx \int_{-\infty}^{\infty} d^{n-1} l dl_n [-l_n^2 - l^2 - m^2 + x(1-x)p^2]^{-2},$$

Now, in polar coordinates in N (now Euclidean) dimensions, choosing $N = 2$,

$$\begin{aligned} \int dl_2 dl_1 &= \int_0^\infty dll \oint_0^{2\pi} d\phi, \\ (l^2 &= l_2^2 + l_1^2), \\ I_2(p^2) &= 2\pi i \int_0^1 dx \int_0^\infty \frac{dll}{[l^2 + m^2 - x(1-x)p^2]^2}, \\ &= \pi i \int_0^1 dx \frac{1}{[m^2 - x(1-x)p^2]^2}, \\ &= \frac{-i}{\pi p^2} \frac{1}{\sqrt{1 - \frac{2m^2}{p^2}}} \ln\left(\frac{\sqrt{1 - \frac{4m^2}{p^2}} - 1}{\sqrt{1 - \frac{4m^2}{p^2}} + 1} \right). \end{aligned}$$

The Wick rotation defines the integral where it is analytical, finding branch cuts by continuation. A branch cut at two-particle threshold is $p^2 \geq 4m^2$. This method is completely general, for any Green's function $G_l(p_1...p_l)$, if one starts with $p_i^2 < 0$ and also $\sum p_i^2 < 0$ (no exceptional momenta). Then, for this case, diagrams can be Wick-rotated diagram by diagram doing all loop diagrams to get all the analytical diagrams.

The generalization of Feynman parameters is (the integral in D dimensions) is

$$\frac{1}{\prod_{i=1}^{A}(D_i+i\epsilon)^{s_i}} = \frac{\Gamma(\sum_{i=1}^{A} s_i)}{\prod_{i=1}^{A}\Gamma(s_i)}\int_0^1 \frac{sx_1 dx_2..dx_A}{[\sum_{i=1}^{A} x_i D_i + i\epsilon]^{\sum s_i}}\delta(1-\sum_{i=1}^{A} x_i)\prod_{i=1}^{A} x_i^{s-1}.$$

It reduces to the previous case and makes it possible to organize the denominator in the loop momenta. Formally, the steps are

- Exchanging k, exchanges x_i (the integral must be convergent for this change, and this is the importance of the dimension regularization).
- Complete the squares.
- Wick rotation ($l_d = il_0$).
- Put in polar coordinates.
- This gives $I_D(p_i,\sigma_i) = i(-1+i\epsilon)^{-s}$, assuming the remaining integral is real.

$$I_D(p_i,\sigma_i,x_i) = \int d\Omega_{D-1}\int_0^\infty \frac{dll^{D-1}}{[l^2+M^2]^{\sum\sigma_i}},$$
$$m^2 = \sum_{i=1}^{s} x_i(\sum_{\sigma=1}^{i} p_i)^2 - (\sum_{i=1}^{i} x_i \sum_{l=1}^{i} p_l)^2$$

The radial integral is

$$\int_0^\infty \frac{dll^{D-1}}{[l^2+M^2]^{\sum\sigma_i}},$$

with a change of variable $y = l^0/\sqrt{M^2}$ and then $y^2 = \eta$, we have finally

$$\int_0^\infty \frac{dll^{D-1}}{[l^2+M^2]^{\sum\sigma_i}} = (M^2)^{\frac{D}{2}-\sum\sigma_i}\frac{1}{2}B(\frac{D}{2},\sum_i \sigma_i - \frac{D}{2}).$$

From diagrams with self-energy, $\sigma_1 \neq 1$. The poles are

- For $D \leq 2\sum \Delta_i, D \leq 0$.
- The radial integral applies any $D \in C$.

The angular integral is evaluated by changing to the cylindrical coordinates in $k \neq 1$ dimensions

$$\begin{aligned} \int d^{k+1}q &= \int_{-\infty}^{\infty} dz_{k+1} \int d^k q, \\ &= \int_{-\infty}^{\infty} dz_{k+1} \int d^q q^{k-1} \int d\Omega_{k-1}, \end{aligned}$$

defining $q_{k+1} = \sqrt{z_{k+1}^2 + q_k^2}$, $\cos\theta_w = \frac{z_{z+1}}{q_{k+1}}$, we get

$$\begin{aligned} \int d^{k+1}q &= \int dq_{k+1} q_{k+1}^k \int_0^\pi d\theta_k \, \sin^{k-1}\theta_k \int d\Omega_{k-1}, \\ &= \int dq_{k+1} q_{k+1}^k \int d\Omega_k, \\ \Omega_{D-1} &= \frac{2\pi^{D/2}}{\Gamma(\frac{D}{2}}, \end{aligned}$$

where $\pi^{\frac{D}{2}} = (\Gamma(\frac{1}{2}))^D$.

Gathering everything together,

$$I_D(p_i, \sigma_i) = i(-i + i\epsilon)^{-\sum \sigma_i} \frac{1}{\pi\gamma(\sigma_i)} \int_0^1 \prod_i dx_i \quad \delta(1 - \sum x_i) x_i^{\sigma_i - 1} \Gamma(\sum \sigma_i - \frac{D}{2}) \pi^{\frac{D}{2}} [M^2(p_i, x_i, m_i)]^{\frac{D}{2} - \sum_i \sigma_i}.$$

The basic integral is obtained doing $s_i = 1$ and defining $\sum_i s_i = s$

$$\begin{aligned} J_{D,s}(q^2, M^2) &= \Gamma(s) \int \frac{d^D k}{(2\pi)^D} \frac{1}{[k^2 + 2kq - M^2 - i\epsilon]^s} \\ &= \frac{i(-1)^s}{(4\pi)^{\frac{D}{2}}} \Gamma(s - \frac{D}{2})(q^2 + M^2 - i\epsilon)^{\frac{D}{2} - s}. \end{aligned}$$

Inside the Feynman parameters integral, one starts with $D < 25$ and taking $D \to 4$ finds isolated poles which will be treat by renormalization.

Example: The ϕ_4^3-Theory

The loop integral is equal to

$$
\begin{aligned}
\Sigma_0(p^2, m^2) &= (+i)(-ig_0)^2 i^2 \frac{1}{2}, \\
&= \int \frac{d^D k}{(2\pi)^D} \frac{1}{k^2 - m^2 - i\epsilon} \frac{1}{(p-k)^2 - m^2 + i\epsilon}, \\
&= \frac{ig_0^2}{\in}{}^1_0 dx J_{D/2}((xp)^2, m^2 - xp^2), \\
&= \frac{(i)^2 g_0^2}{2(4\pi)^{\frac{D}{2}}} \Gamma(2 - \frac{D}{2}) \int_0^1 \Big[m^2 - x(1-x)p^2 - i\epsilon \Big].
\end{aligned}
$$

Separating the poles, which are the ultraviolet correction, one defines $\epsilon = 2 - \frac{D}{2}$ and $\Gamma(\epsilon) \to$ pole as $\epsilon = 0$. The expansion of (7.2.1) and isolating the pole and the infinite yields

$$
\Sigma_0(p^2, m^2) = -\frac{g_0^2}{2(4\pi)^2}(1 - \epsilon \ln 4\pi + ...)(\frac{1}{\epsilon} - \gamma_E ...)(\int_0^1 dx - \epsilon \int_0^1 dx \ln ...),
$$

where we neglect terms that vanish when $\epsilon \to 0$ and almost everything cancels. Problems with this equation is that this pole, however, is divergent for $D = 4$ and there is no scale for logarithm, necessitating the renormalization.

7.3 Terminology for Renormalization

- 1PI diagrams: either vertices or loop diagrams, all ultraviolet divergences are 1PI divergences.

- Notation: $\Gamma_N = -i\gamma_N$, where for $N > 2$ it is just the sum of 1PI diagram.

- For the special case N=2, one has $\Gamma = p^2 - m^2 - i\gamma_2 = p^2 - m^2 - i\Sigma(p^2, m^2)$.

- The Green's function is $G_2 = \frac{i}{\Gamma_2} = \frac{i}{p^2 - m^2 - \Sigma(p^2, m^2)}$. The pole in this function is the physical mass but not necessary the same of the Feynman propagators.

7.4 Classification of Diagrams for Scalar Theories

For a D-Dimensional ϕ^P theory, one has

- L - loop.
- E - external lines
- N - lines
- DL - number of integrals

Looking for superficial overall degree of divergence, all loop momenta goes to infinity at once, one has:

$$\gamma_E = \sum_{1PI} \in \prod_{2-1 \text{ loops}}^{L} d_{kl}^D \prod_{k=1}^{N} \frac{1}{[q^2]}$$

The superficial degree of divergence is

$$\omega(\Gamma_E) = DL - 2N.$$

- $\omega(\Gamma_E) < 0$, superficially convergent,
- $\omega(\Gamma_E) > 0$, superficially power divergent, $\omega(\Gamma_E) = 0$, superficially log divergent.

One fixes D, chooses p (power of the theory and then ω does not depend on the powers of the diagram), and then finds them in function of E and V,

$$L = N - V - 1,$$
$$pV = 2N + E.$$

Together with (7.4.1), one has for each diagram the algebraic function,

$$\omega(\Gamma_E) = D - E(\frac{D-2}{2}) + V(\frac{p}{2}(D-2) - D),$$

where only the last term depends on the order. There are then three cases,

1. $p(\frac{D-2}{2}) - D < 0$, better convergence as order increase (super-renormalizable), for example for $p = 3$ and $d = 4$.
2. $p(\frac{D-2}{2}) - D > 0$, convergence gets worse as V increases (non-renormalizable).

3. $p(\frac{D-2}{2}) - D = 0$, $\omega(\Gamma_E) \geq 0$ only for finite number of E (renormalizable).

The dimensionality of the Lagrangian is then calculated as

$$\mathcal{L} = \frac{1}{2}\mathcal{L}_{KG} - \frac{g}{p!}\phi^p,$$

in D dimensions (natural units) one has

$$[\mathcal{L}] = 1,$$

$$[\partial x] = [m] = 1,$$

$$[\phi] = \frac{D-2}{2},$$

$$[g] = D - p(\frac{D-2}{2}) = D - p[\phi].$$

In terms of ω:

$$\omega(\Gamma_E) = D - E[\phi] - V[g],$$

where

1. $[g] > 0$ superrernomalizable (ϕ_4^3).
2. $[g] = 0$ superrernomalizable (ϕ_6^3, ϕ_4^4).
3. $[g] > 0$ superrernomalizable (ϕ_6^4).

Now one makes the D-dependence of g explicitly introducing a new mass scale M,

$$\mathcal{L} = \frac{1}{2}\mathcal{L}_{KG} - \frac{gM^\epsilon}{p!}\phi^p,$$

where for $D = 4$ one has $[g] = [M^\epsilon] = mass^\epsilon$, $\epsilon = 2 - \frac{D}{2}$.

Other example, for a 6-theory, results in $\frac{-gM^\epsilon}{3!}\phi_6^3$, where $\epsilon = 3 - \frac{D}{2}$. For a gauge theory, $\epsilon = 2 - \frac{D}{2}$.

7.5 Renormalization for ϕ_4^3

From (7.2.1), defining $g_\mu^\epsilon = g_0$,

$$\begin{aligned}\Sigma_0(p^2, m^2) &= \frac{g_0^2}{2(4\pi)^2}[\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \int_0^1 dx \ln (m^2 - x(1-x)\phi^2 - i\epsilon)...], \\ &= \frac{g_0^2}{2(4\pi)^2}[\frac{1}{\epsilon} - \gamma_E + \ln 4\pi - \int_0^1 dx \ln (\frac{m^2}{M^2} - x(1-x)\frac{p^2}{M^2} - i\epsilon) + ...]\end{aligned}$$

where M is the renormalization mass (and an arbitrary constant) and the rest of the equation vanish for $\epsilon \to 0$. The logarithm function is dimensionless but the ultraviolet divergence is

- Additive to finite terms.
- Independent of p (it is local, correspond to a well-defined point).
- Simplly $\frac{g^2}{s(4\pi)^2}\frac{1}{\epsilon}$.

These proprieties allow us to remove the divergence by renormalization which is the modification of the Lagrangian with the new local terms. The counterterms are designed to cancel poles in ϵ. The polynomials in p corresponds to derivatives of the field (which will appear in the general cases). One then defines $\mathcal{L}_{ren} \to \mathcal{L}_{class} + \mathcal{L}_{ctr}$, where $\mathcal{L}_{ctr} = -\frac{1}{2}\delta m^2 \phi^2$ and $\delta m^2 = -\frac{g^2}{2(4\pi)^2}(-\frac{1}{\epsilon} + c_m)$, and this last part is the sum of poles plus an arbitrary part.

The new terminology is given by

- δm^2 os the mass shift.
- c_m is the constant that defines the renormalization scheme.

The new lagrangian, $\mathcal{L}_{ctr}$, is treated as a new vertex in perturbation theory, presented as the figure 7.2, and it can be work out in the momenta space.

$$= -i(2\pi)^4\delta^4(p - p')\delta m^2.$$

Computing Γ^2 in the order g^2 one has

$$\Gamma_2 = \frac{g}{2(2\pi)^4}[c_m - \gamma_E + \ln 4\pi - \int_0^1 dx \ln (\frac{m^2}{M^2} - x(1-x)\frac{p^2}{M^2} - i\epsilon)],$$

resulting that $\epsilon \to 0$ when $D \to 4$.

The classification of the theory, depending on c_m, is given by

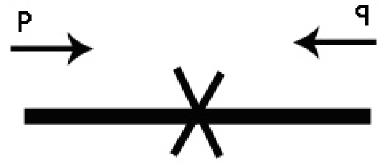

Figure 7.2: New vertex for renormalization.

- $c_m = 0$, minimal subtraction (MS),
- $c_m = \gamma_E - \ \ln 4\pi$, modified minimum subtracted ($\bar{M}S$),
- c_m chosen that for $m^2 = m^2_{phy}$, $\Gamma_2[g^2] = 0$, is the on-shell momentum subtracted scheme (MOM).

One can also add counterterms for tadpoles of ϕ^3_4:

$$= -\frac{ig}{2(4\pi)}^2 \Big(\frac{4\pi M^2}{m^2}\Big)\Gamma(\epsilon - 1)m^2,$$

where the counterterm is $-i\tau$, and it cancels the tadpole completely leaving no finite remainder treatment of tadpole.

Summarizing, using the fact that only two 1PI diagrams are ultraviolet divergent, no other diagrams are superficially. We then write

$$\mathcal{L}_{ren}(\phi^3_4) = \frac{1}{2}\Big((\partial_\mu\phi)^2) - m^2_R\phi^2\Big) - \frac{g_R M^\epsilon}{3!}\phi^3,$$

Adding R in the mass of the original Lagrangian gives $-\frac{1}{2}\delta m^2 c_m \phi^2 - \tau\phi$. This is the renormalization Lagrangian, yielding a finite perturbation theory (but dependent of c_m).

A final remark is that one must combine perturbation theory with physical input, and for this theory, measuring the physical mass of the particle gives $p^2_0 = m^2_{phy}$ where $\Gamma_2 = p^2 - M^2_R - \sum(p^2, M^2_R, g_R, M, c_m) = 0$. When $p^2 = m^2_{phy}$, $G(p^2) = \frac{1}{\Gamma_2}$ goes to infinity. The experimental input is the weight ϕ-part, and demands $\Gamma(p^2 - p^2_0) = 0$,

$$\begin{aligned}\mathcal{L}_{ren} &= \frac{1}{2}((\partial_M\phi)^2 - m^2_R\phi^2) - g\frac{M^\epsilon\phi}{3!}\phi^3 - \frac{1}{2}\delta m^2\phi^2, \\ &= \frac{1}{2}((\partial_M\phi)^2 - m^2_0\phi^2) - \frac{g_0}{3!}\phi^3.\end{aligned}$$

The final important result is

$$\mathcal{L}_{ren} = \mathcal{L}_{barl}(m_0, g_0),$$

where from

1. LHS, the perturbation theory is in term of $m_R^2, g_R^2 = g_0 M^\epsilon = m_0^2 + \delta m^2$, and it is finite order by order in perturbation theory (every divergent diagram is canceled).

2. RHS, the perturbation theory only depends on two parameters.

From item 1, one picks c_m and a value for M, then the Green's function are ultraviolet finite but depend on c_m and M.

From item 2, choices on c_m and M only affects the relation between m_p and m_{phy}. Once one knows m_R for some M, it is possible to calculate all remaining Green's function for the S-matrix.

7.6 Guideline for Renormalization

This steps are for $\lambda\psi^3$, $d = 6$, $\omega(\Gamma) = 6 - 2N$, however they give the route for farther process of renormalization,

1. The Lagrangian is $\mathcal{L} = \frac{1}{2}\partial_\mu\psi\partial_\mu\psi - \frac{1}{2}m^2\psi^2 + \frac{\lambda}{3!}\psi^3$. One has two graphics and since $\omega(\Gamma) > 0$, they might be divergent.

2. Introduce a new coupling $\lambda \to \lambda_\mu^{\delta(N,d)}$, in the new dimension $N \sim d + \epsilon$. In this case $\delta = -\frac{1}{2}(N - d) = \frac{1}{2}N + 3 = \epsilon$, thus $\epsilon = 3 - \frac{n}{2}$ and one rewrites $\lambda \to \lambda_\mu^\epsilon$.

3. From drawing the diagram one has
$$= \frac{\lambda^2\mu^{2\epsilon}}{(2\pi)^N}\int \frac{d^n l}{(l - m^2)((p - l)^2 - m^2)}.$$

4. Wick rotating (the momentum becomes imaginary):
$$= \frac{\lambda^2\mu^{2\epsilon}}{(2\pi)^N}\int \frac{d^N l}{(-l^2 - m^2)(-(p - l)^2 + m^2)}.$$

5. Using the Feynman relation $\frac{1}{AB} = \int_o^1 dx \frac{1}{(xA + (1-x)B)^2}$, gives
$$= \frac{(\lambda\mu^\epsilon)}{(2\pi)^N}\int_0^1 dx \int \frac{d^N l}{(2\pi)^N}\frac{1}{x(l^2 + m^2 + (1 - x))((l - p) + m^2)^2}.$$

6. With the variable change $l' = l - lp(1-x)$ to get read of the l terms when squaring l', yields

$$= \frac{\lambda^2 \mu^{2\epsilon}}{(2\pi)^N} \int_o^2 dx \int \frac{d^N l}{(l'^2 + m^2 + p^2(1-x)x)^2}.$$

7. The solution of this integral is given by

$$\int \frac{d^N l}{(2\pi)^N} \Big(\frac{1}{(l^2 - M^2)^A} \Big) = \frac{\Gamma(A - \frac{n}{2})}{(2\pi)^{\frac{N}{2}} \Gamma(A)} \frac{1}{(M^2)^{A - \frac{1}{2}}}.$$

Resulting in

$$= \frac{\lambda^2 \mu^{2\epsilon}}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma(2 - \frac{n}{2})}{2} \int_o^1 dx \frac{1}{(m^2 + p^2 x(1 - x(1-x))^{2-n/2}}.$$

8. Performing $\frac{n}{2} = 3 - \epsilon$, results on $\Gamma(2 - \frac{N}{2}) = \Gamma(-1 + \epsilon)$. The integral is then

$$= \frac{\lambda^2 \mu^{2\epsilon}}{(2\pi)^{\frac{1}{2}}} \frac{\Gamma(\epsilon - 1)}{2} \int_0^1 dx \frac{1}{(m^2 + p^2 x(1-x))^{\epsilon - 1}}.$$

9. The next step is to subtract the divergences using $\Gamma(\epsilon - N) = \frac{1}{\epsilon} - \gamma_E$, resulting on

$$= \frac{\lambda^2 \mu^{2\epsilon}}{(2\pi)^{\frac{N}{2}}} \frac{\Gamma(\epsilon - 1)}{2} \int_0^1 dx \frac{m^2 + p^2 x(1-x)}{(m^2 + p^2 x(1-x))^{\epsilon}}.$$

10. From the expansion $a^\epsilon = 1 + \epsilon \ln a$ and some algebra finally gives the divergent term in first order,

$$= \frac{\lambda^2}{(2\pi)^3} \frac{1}{2} (m^2 + \frac{4}{3} p^2) \frac{1}{\epsilon} + \mathcal{O}(\epsilon).$$

The counter-terms are always proportional to m^2, p^2 and λ.

Chapter 8

Sigma Model

Part II

Topological Field Theories

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